MATRIX GROUPS OF THE SECOND KIND

BY

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A group, G, of matrices with entries from the field of complex numbers is said to be "of the second kind" if each matrix has real character, but G is not similar to a group of matrices with real entries. The single faithful irreducible representation of the quaternion group provides an example of such a group of matrices.

Two classical results are strengthened by Theorem 1 below: The first of these asserts that every non-trivial irreducible representation of a (finite) group of odd order involves complex characters. (We deal exclusively here with finite groups, and representations over the field of complex numbers.) Theorem 1 extends to the more general case of a group whose elements of odd order form a subgroup. The second classical result asserts that every matrix group of the second kind has even degree. Theorem 1 puts a constraint on this degree, leading to the easy corollary that a group whose order is not divisible by four cannot have an irreducible representation of the second kind.

Theorem 2 complements Theorem 1 by providing a set of circumstances under which we may assert that a group, G, *does* have a representation of the second kind.

I would like to acknowledge my indebtedness to Dr. G. de B. Robinson who, as my supervisor, brought these problems to my attention, and to the referee, whose suggestions made Theorem 1 possible, by supplying an easy proof of the corollary.

THEOREM 1. *Let G be a group whose elements of odd order form a subgroup, and suppose that $\rho(G)$ is an irreducible representation, of the second kind, of G. Then the order of G is divisible by twice the degree of $\rho(G)$.*

Proof. We may, without loss, assume that $\rho(G)$ is a faithful representation of G.

Let N be the subgroup of G which consists of all the elements of odd order in G. $N \lhd G$, and we may invoke Clifford's Theorem in considering $\rho(G) \downarrow N$, which has irreducible components $\sigma_i(N)$, with common multiplicity n. All of the $\sigma_i(N)$ are in the same family of irreducible representations of N.

Let f be the degree of $\rho(G)$, and suppose that the theorem is false, so that $2f$ does not divide $|G|$ (f, of course, does). Let P be a Sylow 2-group of G. It is trivial to show that G is a semi-direct product NP. Suppose that $|P| = 2^k$. Then $f = 2^k s$, with s odd. Suppose also that each $\sigma_i(N)$ has degree t (they are all in the same family of representations of N) and that z different irreducible representations of N appear in $\rho(G) \downarrow N$. Then $tzn = f = 2^k s$. Further,

Received July 28, 1969.