SEQUENCE MIXING AND α-MIXING

BY

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1. Introduction

Let $(\Omega, \mathfrak{A}, m)$ be a probability space and let τ be a bimeasurable, invertible transformation mapping Ω onto Ω. All sets discussed throughout will be assumed to be elements of $\mathfrak{A}$. τ is measure-preserving if $m(\tau A) = m(A)$ for all A, it is ergodic if

$$\lim (1/n) \sum_{j=0}^{n-1} m(\tau^j A \cap B) = m(A)m(B) \quad \text{for all } A \text{ and } B,$$

it is weak mixing if

$$\lim (1/n) \sum_{j=0}^{n-1} | m(\tau^j A \cap B) - m(A)m(B) | = 0 \quad \text{for all } A \text{ and } B,$$

and strong mixing if

$$\lim m(\tau^n A \cap B) = m(A)m(B) \quad \text{for all } A \text{ and } B.$$

Since weak mixing already implies that $\lim m(\tau^n A \cap B) = m(A)m(B)$, except possibly along a sequence of asymptotic density zero (which may depend on A and B), it might be supposed that there is no room between weak mixing and strong mixing. At a symposium on ergodic theory held at Tulane University in October 1961, one of the authors proposed the notion of sequence mixing. τ is sequence mixing if for every A with $m(A) > 0$ and every infinite sequence of integers $\{k_n\}$ we have $m(\bigcup \tau^{k_n} A) = 1$. It is trivial to verify that strong mixing implies sequence mixing but for a number of years it remained an open question whether the converse holds. Recently Friedman and Ornstein, [2] showed that this is not the case. They define a transformation τ to be α-mixing for $\alpha \in (0, 1)$ if

$$\liminf_n m(\tau^n A \cap B) \geq \alpha m(A)m(B) \quad \text{for all } A \text{ and } B,$$

and show that for every $\alpha \in (0, 1)$ there exist transformations which are α-mixing but not $(\alpha + \varepsilon)$-mixing for any $\varepsilon > 0$. Thus we may suppose that for every $\alpha \in (0, 1)$ there exists an α-mixing transformation and sets A and B with $m(A) > 0$, $m(B) > 0$ and $\liminf_n m(\tau^n A \cap B) = \alpha m(A)m(B)$.

In this paper we construct a transformation which is sequence mixing but not α-mixing for any $\alpha \in (0, 1)$. It follows from the lemma below that α-mixing implies sequence mixing and it follows from [1] that sequence mixing implies weak mixing. Therefore α-mixing is strictly between weak and strong mixing. Also Friedman [3] gives an example of a weak mixing transformation T such that for some set A with $0 < m(A) < 1$ we have

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