RANK 3 SUBGROUPS OF ORTHOGONAL GROUPS

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1. Introduction

The projective commutator subgroup of an orthogonal group over a finite field K of odd characteristic, $P\Omega_n(K)$ (where $n \geq 5$), is known to be a rank 3 permutation group on the one-dimensional singular subspaces of the underlying vector space [6]. In this paper, we prove the following theorem:

MAIN THEOREM. *Let $H < P\Omega_n(K)$, $5 \leq n \leq 7$, K a finite field of odd characteristic. Suppose H is a rank 3 permutation group on the one-dimensional singular subspaces of the underlying n-dimensional orthogonal space. Then $H = P\Omega_n(K)$.*

This result is analogous to a result of Higman and McLaughlin on rank 3 subgroups of symplectic and unitary groups for dimensions $4 \leq n \leq 8$ [3], later improved to include dimensions $n > 8$ by Perin [4]. Unfortunately, we were not able to apply Perin's method in this paper. However, we do prove some lemmas in Sections 3, 4, and 6 which hold for all dimensions $n \geq 5$, and which may be of independent interest. In addition, we make some remarks in Section 8, explaining why the question in the main theorem does not make sense for smaller dimensions.

2. Siegel elations

Let $B(x, y)$ be the nondegenerate symmetric bilinear form defining the orthogonal space. If $B(x, x) = 0$ and x is not zero, we say x is *singular*. If all of the vectors of some nonzero subspace are singular or zero we say the subspace is singular. The set of all vectors y such that $B(x, y) = 0$ for all x in some subset S of vectors is called $S^\perp$ ("S perp"). With these notations, we can define a Siegel transformation $\rho(x, u)$ in $\Omega_n(K)$, and its image in $P\Omega_n(K)$, the Siegel elation $r(x, u)$. (A Siegel elation is an elation on the hyperplane $x^\perp$.)

A Siegel transformation, $\rho(x, u)$, (where x is singular, and $B(x, u) = 0$) sends vectors v in $x^\perp$ to $v + B(v, u)x$. Tamagawa [6] shows that such a transformation can be extended in only one way to an element (also called $\rho(x, u)$) of the orthogonal group. Precisely, if y is a singular vector such that $B(x, y) = 1$ and $B(u, y) = 0$ (such a vector y can always be found), then $\rho(x, u)$ sends y to $y - u - Q(u)x$ (where $2Q(u) = B(u, u)$).

We define a *point* to be a one-dimensional subspace and a *line* to be a 2-dimensional subspace.

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