ON ISOCLINIC CLOSURE: CORRECTION TO "ELLIPTIC SPACES IN GRASSMANN MANIFOLDS"[1]

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Let $\mathbf{F}$ be a field $\mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{K}$ (quaternion), $\mathbf{F}^k$ the positive definite left unitary vector space of dimension k over $\mathbf{F}$, and $\mathbf{G}_{n,k}(\mathbf{F})$ the Grassmann manifold of n-dimensional $\mathbf{F}$-subspaces of $\mathbf{F}^k$ with its usual structure as a Riemannian symmetric space. If W is a subspace of $\mathbf{F}^k$ then π_W denotes the orthogonal projection $\mathbf{F}^k \to W$. Subspaces B, B' of the same dimension in $\mathbf{F}^k$ are *isoclinic* if $\pi_B\colon B' \to B$ is an $\mathbf{F}$-unitary similarity. For example, the connected totally geodesic submanifolds $\mathbf{B}$ in $\mathbf{G}_{n,k}(\mathbf{F})$ such that any two distinct elements of $\mathbf{B}$ have zero intersection as subspaces of $\mathbf{F}^k$, have the property that the elements of $\mathbf{B}$ are pairwise isoclinic [2, Theorems 2 and 4], [3, Theorem 2]. In [2] and [3] one finds a complete classification of all such submanifolds $\mathbf{B}$.

After writing out that classification, Wolf considered an arbitrary subset $\mathbf{A}$ of $\mathbf{G}_{n,k}(\mathbf{F})$ whose elements are pairwise isoclinic, and in [3, Section 7] he claimed to define an operation of "isoclinic closure" enlarging $\mathbf{A}$ to a totally geodesic submanifold $\mathbf{A}_* = \mathbf{B} \subset \mathbf{G}_{n,k}(\mathbf{F})$ of the type described above. That isoclinic closure operation depended in an essential manner on the following property:

(*) If $X, B, B' \in \mathbf{G}_{n,k}(\mathbf{F})$ are pairwise isoclinic with $B \neq B'$ then $Z = \pi_{B\oplus B'}(X)$ either is 0 or is n-dimensional and isoclinic to X.

If $k = 2n$ then (*) follows from the Hurwitz equations; see [2, Theorem 1]. Professor Y.-C. Wong [4; Chapter III, Section 11] and Daniel Shapiro (unpublished) gave examples showing that (*) fails for $(n, k) = (2, 6)$, and thus fails whenever n is even and $k \geq 3n$.

Here is a counterexample to (*) for $k = 2n + 1$, thus for all (n, k) with $k > 2n$. Let $\{e_1, \ldots, e_n; f_1, \ldots, f_n; u\}$ be an orthornormal basis of $\mathbf{F}^{2n+1}$ and define the linear spans

$$B = \{e_1, \ldots, e_n\}_{\mathbf{F}} \quad \text{and} \quad B' = \{e_1 + f_1, \ldots, e_n + f_n\}_{\mathbf{F}},$$

and

$$X = \{x_1, \ldots, x_n\}_{\mathbf{F}} \quad \text{where } x_1 = e_1 + f_1 + 2\sqrt{2}\,u$$

and

$$x_j = e_j - 3f_j \quad \text{for } 2 \leq j \leq n.$$

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