UNIQUENESS OF A CLASS OF FUCHSIAN GROUPS[1]

BY

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1. Let G be a fuchsian group of Moebius transformations acting on the upper half-plane $\mathbf{H}$, i.e., G is a discrete subgroup of $LF(2, \mathbf{R})$. As usual, we treat G as though it were a matrix group. Let G contain translations. We consider the parameter

$$c_0(G) \equiv c_0 = \min \{|c| \neq 0: (a, b: c, d) \in G\}. \tag{1.1}$$

It is well known that the minimum is attained and that $c_0 > 0$. Under certain circumstances the value of c_0 characterizes G up to conjugacy.

Since G contains translations, it will contain a smallest translation $z \to z + \lambda$, $\lambda > 0$. If $\lambda = 1$ we say G is *normalized.* Any group G can be normalized by conjugation with $\theta = (\lambda^{-1/2}, 0: 0, \lambda^{1/2})$ and we write

$$G^* = \theta G \theta^{-1} \tag{1.2}$$

for the normalized group. The notation K^* means that K is normalized. Obviously $c_0(G^*) = \lambda c_0(G)$.

Among the well-known groups in this class are the Hecke groups H_q. Here

$$H_q = \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad 3 \le q \le \infty, \tag{1.3}$$

where

$$\lambda_q = 2 \cos \frac{\pi}{q}, \; 2 \le q < \infty; \; \lambda_\infty = 2.$$

The Hecke groups are included in the more general class

$$H_{p,q} = \left\langle \begin{pmatrix} 1 & \lambda_p + \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -\lambda_p \end{pmatrix} \right\rangle, \quad 2 \le p \le q \le \infty, \, p + q > 4; \tag{1.4}$$

in fact $H_q = H_{2,q}$. (There is no group $H_{2,2}$; see the lines following (2.6).) We shall see (Section 2) that

$$c_0(H_q) = c_0(H_{p,q}) = 1; \tag{1.5}$$

hence

$$c_0(H_q^*) = \lambda_q, \; c_0(H_{p,q}^*) = \lambda_p + \lambda_q. \tag{1.6}$$

It is known [3] that $H_{p,q}$ is the free product of a cyclic group of order p and one of order q when $p, q < \infty$.

In this paper all conjugacies will be over $SL(2, R)$.

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