THE PRODUCT OF CONSECUTIVE INTEGERS IS NEVER A POWER

BY

P. ERDÖS AND J. L. SELFRIDGE

We dedicate this paper to the memory of our friends H. Davenport, Ju.V. Linnik, L. J. Mordell, L. Moser, A. Rényi and W. Sierpiński, all of whom were alive when we started our work in 1966 at the University of Illinois at Urbana.

0. Introduction

It was conjectured about 150 years ago that the product of consecutive integers is never a power. That is, the equation

$$(n + 1)\cdots(n + k) = x^l \tag{1}$$

has no solution in integers with $k \geq 2$, $l \geq 2$ and $n \geq 0$. (These restrictions on k, l and n will be implicit throughout this paper.) The early literature on this subject can be found in Dickson's history and the somewhat later literature in the paper of Obláth [5].

Rigge [6], and a few months later Erdös [1], proved the conjecture for $l = 2$. Later these two authors [1] proved that for fixed l there are at most finitely many solutions to (1). In 1940, Erdös and Siegel jointly proved that there is an absolute constant c such that (1) has no solutions with $k > c$, but this proof was never published. Later Erdös [2] found a different proof; by improving the method used, we can now completely establish the old conjecture. Thus we shall prove:

THEOREM 1. *The product of two or more consecutive positive integers is never a power.*

In fact we shall prove a stronger result:

THEOREM 2. *Let k, l, n be integers such that $k \geq 3$, $l \geq 2$ and $n + k \geq p^{(k)}$, where $p^{(k)}$ is the least prime satisfying $p^{(k)} \geq k$. Then there is a prime $p \geq k$ for which $\alpha_p \not\equiv 0 \pmod{l}$, where α_p is the power of p dividing $(n + 1)\cdots(n + k)$.*

Theorem 2 implies Theorem 1, since it is easy to see that $(n + 1)(n + 2)$ is never an lth power and if $n \leq k$ then by Bertrand's postulate the largest prime factor of $(n + 1)\cdots(n + k)$ divides this product to exactly the first power. Moreover, this shows that in proving Theorem 2 it will suffice to assume $n > k$.

One could conjecture the following strengthening of Theorem 2: if $k \geq 4$ and $n + k \geq p^{(k)}$, then there is at least one prime greater than k which divides

Received May 24, 1974.