REPRESENTING MEASURES IN COMPACT GROUPOIDS

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1. Introduction

The objective of this paper is to describe an abstract theory of representing measures. To do this we consider a compact topological groupoid, i.e., X is compact Hausdorff and $\cdot : X \times X \to X$ is continuous. $(X, \cdot)$ is commutative if $x \cdot y = y \cdot x$ for every $x, y \in X$, and $(X, \cdot)$ is medial if $(w \cdot x) \cdot (y \cdot z) = (w \cdot y) \cdot (x \cdot z)$ for every $w, x, y, z \in X$. Observe that if X is a compact convex subset of a locally convex topological vector space with $\cdot$ as the midpoint function, then $(X, \cdot)$ is commutative and medial. Throughout this paper we shall refer to such a set as simply a compact convex set. With this example in mind define a real valued function f on a compact groupoid $(X, \cdot)$ to be convex if for every $x, y \in X$, $f(x \cdot y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$. Let $C(X)$ denote the continuous real valued functions on X and let C denote the continuous convex functions on X. An element $x \in X$ is called an idempotent if $x \cdot x = x$ and we call the set of all idempotents of X the core of X and denote it by core X. A class of functions K on a set S is said to separate points if for every $x, y \in S$ with $x \neq y$, there exists $f \in K$ such that $f(x) \neq f(y)$. We shall say that K is totally separating if $x, y \in S$ and $f(x) \geq f(y)$ for every $f \in K$ implies that $x = y$. $(X, \cdot)$ is said to be strongly separated by its convex functions if C separates the points in X and C is totally separating on core X. If $(X, \cdot)$ is a compact medial groupoid that is strongly separated by its convex functions, then $(X, \cdot)$ is called a compact mean space.

When X is a compact Hausdorff space then we shall let $\Omega(X)$ denote the regular Borel probability measures on X. Since $\Omega(X)$ is exactly those regular signed Borel measures μ in the closed unit ball of $C(X)^*$ such that $\int 1 \, d\mu = 1$, $\Omega(X)$ is weak* compact. If $(X, \cdot)$ is a compact groupoid and $\mu, \nu \in \Omega(X)$, then

$$l(f) = \int f(x \cdot y) \, d\mu(x) \times \nu(y)$$

defines a norm one linear functional l on $C(X)$ such that $l(1) = 1$. Hence $l(f) = \int f \, d\phi$ for some $\phi \in \Omega(X)$. We shall denote the measure ϕ by $\mu * \nu$. $\mu * \nu$ is called the convex convolution of μ with ν. Now define a map $S: \Omega(X) \to \Omega(X)$ by $S(\mu) = \mu * \mu$ for every $\mu \in \Omega(X)$. Since $\cdot$ is continuous, it is easily verified that S is weak* continuous. If $\mu \in \Omega(X)$ and $x \in X$, then we say that μ represents x if for every $f \in C(X)$,

$$\lim_{n \to \infty} \int f \, dS^n(\mu) = f(x).$$

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