ESSENTIALLY (G_1) OPERATORS AND ESSENTIALLY CONVEXOID OPERATORS ON HILBERT SPACE

BY

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Introduction

Let H be a separable Hilbert space and let $\mathscr{B}(H)$ be all operators (continuous linear transformations) from H into H. Let π be the quotient map from $\mathscr{B}(H)$ onto the Calkin algebra $\mathscr{B}(H)/\mathscr{K}$, where $\mathscr{K}$ denotes all compact operators in $\mathscr{B}(H)$. $T \in \mathscr{B}(H)$ is essentially normal, essentially hyponormal, essentially G_1, or essentially convexoid if $\pi(T)$ is normal, hyponormal, G_1, or convexoid in $\mathscr{B}(H)/\mathscr{K}$, respectively. Denote each of the above sets in $\mathscr{B}(H)$ by $e(\mathscr{N})$, $e(\mathscr{H})$, $e(\mathscr{G})$, and $e(\mathscr{C})$, respectively, where $\mathscr{N}$ is the set of all normal operators on H, $\mathscr{H}$, is the set of all hyponormal operators on H, $\mathscr{G}$ is the set of all operators on H satisfying growth condition G_1 (i.e. $\|(T - z)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$ where $\sigma(T)$ denotes the spectrum of T), and $\mathscr{C}$ is the set of all convexoid operators on H (i.e., the convex hull of the spectrum of T, co $\sigma(T)$, is equal to the closure of the numerical range of T, $\overline{W(T)}$). The spectral properties of essentially G_1 operators and essentially convexoid operators are discussed in [9]. Along with ways of constructing nontrivial examples, section one contains several elementary facts about elements in the Calkin algebra and some of the basic properties of essentially G_1 operators and essentially convexoid operators. The main results of the second section are: (1) $e(\mathscr{N})$ is a closed nowhere dense subset of $e(\mathscr{H})$, (2) $e(\mathscr{H})$ is a closed nowhere dense subset of $e(\mathscr{G})$, (3) $e(\mathscr{G})$ is a closed nowhere dense subset of $e(\mathscr{C})$, and (4) $e(\mathscr{C})$ is a closed nowhere dense subset of $\mathscr{B}(H)$.

I. Basic properties and examples

For each $T \in \mathscr{B}(H)$ let $\sigma_e(T)$ denote the essential spectrum of T, i.e., $\sigma_e(T)$ is the set of all complex numbers λ such that $\pi(T) - \lambda$ is not invertible in the Calkin algebra. The proof of the following remark is straightforward.

Remark 1. If $T = A \oplus B$ on $H \oplus H$, then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(B)$.

THEOREM 1. *If $A, B \in \mathscr{B}(H)$, then* $\|\pi(A \oplus B)\| = \text{Max}\,\{\|\pi(A)\|, \|\pi(B)\|\}$.

Theorem 1 is an immediate consequence of Remark 1 and the fact that the norm of a self-adjoint element of a B^*-algebra is equal to its spectral radius.

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