SEMISIMPLICITY OF 2-GRADED LIE ALGEBRAS

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1. Introduction

In a graded Lie algebra, the defining commutator identities involve signatures depending on the parity of the degrees. Being concerned only with the structural significance of these signatures, we consider only Z_2-gradings, and we call them 2-gradings.

With regard to the notion of semisimplicity, already a casual exploration of 2-graded Lie algebras brings up a surprise. Over any field of characteristic 0, there are 2-graded Lie algebras of arbitrarily high dimension which are *simple*, in the sense of having no proper ideals, but not *semisimple* in the module theoretic sense. In fact, we shall see in Section 5 that the most conventional and natural construction leads to precisely such algebras. Thus, already the first question about the existence of semisimple 2-graded Lie algebras other than the ordinary ones, in which the component of degree 1 is (0), does not have an immediate answer. However, we hasten to add that such 2-graded Lie algebras do exist.

In Section 2, we give a precise description of our setting, and we discuss the basic special features of the universal enveloping algebra of a 2-graded Lie algebra. In Section 3, we deal with the elementary facts concerning semisimple graded modules for 2-graded rings, which we need in Section 4 for proving that the direct sum of semisimple 2-graded Lie algebras is semisimple. Owing to the lack of an intrinsic characterization of semisimple 2-graded Lie algebras, this result, at present, is not as trivial as it ought to be. The other results of Section 4 aim at a reduction of the structure theory to the theory of ordinary semisimple Lie algebras and their representations. In fact, it seems to be most appropriate and promising to regard semisimple 2-graded Lie algebras as a superstructure to be built over classical Lie algebra theory. From this point of view, our only isolated specimen, exhibited in Section 6, appears to be of basic significance, resting upon and extending the representation theory of $sl(2)$.

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