A CLASS OF DIFFERENTIAL INEQUALITIES IMPLYING BOUNDEDNESS

BY

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Let B denote the class of bounded functions $w(z) = w_1 z + w_2 z^2 + \cdots$ regular in the unit disc U for which $|w(z)| < 1$. If $g(z) \in B$, then by using the Schwarz lemma we can show that the function $w(z)$ defined by $w(z) = z^{-1/2} \int_0^z g(t)t^{-1/2}\, dt$ is also in B. Writing this result in terms of derivatives we have

$$(1) \qquad |\tfrac{1}{2}w(z) + zw'(z)| < 1,\ z \in U \Rightarrow |w(z)| < 1,\ z \in U.$$

All of the inequalities considered in this paper hold uniformly in the unit disc U, and in what follows we will omit the condition $z \in U$. Furthermore, if we let $h(u, v) = \frac{1}{2}u + v$ we can write (1) as

$$(2) \qquad |h(w(z), zw'(z))| < 1 \Rightarrow |w(z)| < 1.$$

In this note we will show that (2) holds for functions $h(u, v)$ satisfying the following definition.

DEFINITION 1. Let H be the set of complex functions $h(u, v)$ satisfying:

(i) $h(u, v)$ is continuous in a domain D of $\mathbf{C} \times \mathbf{C}$,
(ii) $(0, 0) \in D$ and $|h(0, 0)| < 1$,
(iii) $|h(e^{i\theta}, ke^{i\theta})| \geq 1$ when $(e^{i\theta}, ke^{i\theta}) \in D$, θ is real and $k \geq 1$.

Examples. It is easy to check that each of the following functions is in H:

$$h_1(u, v) = \alpha u + v \text{ where } \alpha \text{ is complex with } \operatorname{Re} \alpha \geq 0, \text{ and } D = \mathbf{C} \times \mathbf{C},$$

$$h_2(u, v) = u^2 + u + v \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_3(u, v) = \tfrac{1}{3}(|u| + |v| + 1) \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_4(u, v) = 2\, v/(1 - u) \quad \text{and} \quad D = [\mathbf{C} - \{1\}] \times \mathbf{C},$$

$$h_5(u, v) = ue^{|v|} \quad \text{and} \quad D = \mathbf{C} \times \mathbf{C},$$

$$h_6(u, v) = u^m v^n \text{ where } m \text{ and } n \text{ are non-negative integers, and } D = \mathbf{C} \times \mathbf{C}.$$

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