MEASURABLE WEAK SECTIONS

BY

G. A. EDGAR

Introduction

If S and T are sets and $p: T \to S$ is a function, then a *section* of p is a function $q: S \to T$ such that $p \circ q = 1_S$. According to the Axiom of Choice, there is a section of p if and only if p is surjective. If $\langle S, \mathscr{F} \rangle$ and $\langle T, \mathscr{G} \rangle$ are measurable spaces (or "Borel spaces"; S is a set and $\mathscr{F}$ is a sigma-algebra of subsets of S) and if $p: T \to S$ is measurable (i.e., $p^{-1}(A) \in \mathscr{G}$ for all $A \in \mathscr{F}$), the question of the existence of a *measurable* section $q: S \to T$ arises. If there is a measure μ on S, it might only be required that $p \circ q = 1$ μ-almost everywhere. We consider below a still less restrictive possibility (called a "weak section").

The existence of measurable sections and related questions have been considered by many mathematicians: a good survey with many references is Parthasarathy's book [20]. Other references not mentioned there include [7], [22, Lemma 4.1, p. 27], [2, Theorem 4, p. 135], [24], [4, Theorem 6], [19, p. 15], [3, Chapter VIII], [17]. All of these papers (except [3]) assume at least that T is metrizable space, most assume that it is also separable. ([24] assumes only that T has a base with cardinal not exceeding the first uncountable cardinal and is hereditarily Lindelof.) In the present paper, we are interested in the situation for "large" spaces T.

If S and T are measurable spaces and $p: T \to S$ is a measurable function, then a map p_* which takes finite measures on T to finite measures on S may be defined by $p_*(\lambda)(B) = \lambda(p^{-1}(B))$. In this situation, the following question has been asked: If μ is a finite measure on S, does there exist a finite measure λ on T such that $p_*(\lambda) = \mu$? This can be viewed as a problem on the extension of a measure to a larger sigma-algebra. (See [18], [15], [1], [27], [13], [14].)

It is known [27], [14] that there is a connection between these two problems. Indeed (compare Theorem 1.2, below), if q is a measurable section of p, then $\lambda = q_*(\mu)$ has the property $p_*(\lambda) = \mu$. In this paper we establish an approximate converse for this result under certain topological conditions. Roughly speaking, we show that if λ_0 is an extreme point of the set of all measures λ with $p_*(\lambda) = \mu$, then λ_0 is of the form $q_*(\mu)$ for some weak section q of p. See the precise statement and proof below (Theorem 1.3).

In the classical case of the problem of sections, the problem is related to two others, namely "selection" and "uniformization," which can be described (somewhat oversimplified) as follows. Let $T' = S \times T$ and let $u: T' \to S$ be

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