GEOMETRICAL PROPERTIES DETERMINED BY THE HIGHER DUALS OF A BANACH SPACE

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1. Introduction

Let X be a Banach space and X^*, X^{**}, X^{***}, and $X^{(4)}$ the successive dual spaces. We denote by J_0, J_1, and J_2 the natural embeddings of X, X^*, and X^{**} into X^{**}, X^{***}, and $X^{(4)}$ respectively. When no confusion can result we shall omit these maps and write, for example, $x \in X^{**}$ to mean $J_0(x) \in X^{**}$.

Among the consequences of a result of Dixmier [5] is the fact that if $x^{**} \in X^{**}$ then

$$\|J_2(x^{**}) - J_0^{**}(x^{**})\| \geq \operatorname{dist}(x^{**}, X).$$

It is easy to verify that

$$J_2(x^{**})|_{X^*} = J_0^{**}(x^{**})|_{X^*}$$

and so if $x^{**}(x^*) = 1$ where $\|x^{**}\| = \|x^*\| = 1$ and $x^{**} \notin X$, then $J_1(x^*)$ has two distinct norming elements in $X^{(4)}$. Since by a famous theorem of James [8], [9] such an x^{**} and x^* must exist if X is not reflexive we have that if X is not reflexive then X^{***} is not smooth and $X^{(4)}$ is not rotund.

It is clear from this formulation of Dixmier's theorem that reflexivity is implied by a condition on X weaker than X^{***} smooth, involving only the behavior of X^* in X^{***} and X in X^{**}. This suggests the possibility of studying geometrical properties of Banach spaces determined by viewing the spaces as subspaces of their second duals. In this paper we define several such properties and explore the connections among them and other geometrical notions.

Section 2 contains various background information. In particular we examine Dixmier's theorem in a little more detail with a view to motivating the definitions which are taken up in the later sections.

Since our properties are defined using the second and third dual spaces, the fact that B (the unit ball of X) is weak* dense in B^{**} (the unit ball of X^{**}) and Helly's theorem are used extensively. Particularly useful is the elegant combination of Goldstine's theorem and Helly's theorem expressed in the principle of local reflexivity [12]. A version of this result due to Dean [3] is stated in Section 2.

One of the principal themes of this work is the exploitation of the observation that there is a dichotomy of geometrical conditions determined by which

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