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A PROBLEM IN THE EXTENSION OF MEASURES

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1. Introduction

The problem referred to in the title is as follows. Suppose $(X, \mathscr{B})$ is a measurable space, $\mathfrak{A}$ a sub-σ-field of $\mathscr{B}$ and μ a probability measure on $\mathfrak{A}$. Can μ be extended to a measure on $\mathscr{B}$?

E. Marczewski has discussed various aspects of this problem in a number of articles, notably [10], [11], [12]. In [10], he constructed an example of a non-separable probability measure μ on a sub-σ-field $\mathfrak{A}$ of the Borel σ-field $\mathscr{B}$ of the unit interval [0, 1]. Plainly μ cannot be extended to a measure on $\mathscr{B}$, which shows that the answer to the problem posed above is, in general, no. Marczewski [11] then asked if a separable measure μ on a sub-σ-field $\mathfrak{A}$ of the Borel σ-field $\mathscr{B}$ of the unit interval could be extended to $\mathscr{B}$. The answer is again no as the following (unpublished) example of Marczewski shows. Take $\mathfrak{A}$ to be the σ-field of meager and comeager Borel subsets of the unit interval and let μ be the measure on $\mathfrak{A}$ which is 0 on meager Borel sets and 1 on comeager Borel sets. So defined, μ is a separable probability measure on $\mathfrak{A}$. But, as is well known, a probability measure on the Borel σ-field of the unit interval sits on a meager Borel set. Hence μ cannot be extended to a measure on the Borel σ-field.

In view of these examples, one is led to reformulate the above problem. To do so we first introduce some definitions. We shall say that a measurable space $(X, \mathscr{B})$ has the *extension property* if for every countably generated sub-σ-field $\mathfrak{A}$ of $\mathscr{B}$ and any probability measure μ on $\mathfrak{A}$, μ can be extended to a measure on $\mathscr{B}$. Of particular interest will be measurable spaces $(X, \mathscr{B})$ such that $\mathscr{B}$ is countably generated and contains singletons. It is known that, if $(X, \mathscr{B})$ is such a measurable space, then X can be metrized in such a way that it becomes a separable metric space and $\mathscr{B}$ is just the Borel σ-field of X. Conversely, if X is a separable metric space, then the Borel σ-field $\mathscr{B}$ of X is countably generated and contains singletons. This leads us to the following definition. We say that a separable metric space X has the *extension property* if $(X, \mathscr{B}_X)$ has the extension property, where, for a metric space Y, $\mathscr{B}_Y$ denotes the Borel σ-field of Y.

The question naturally arises if every separable metric space has the extension property. We do not know if a separable metric space without the extension property can be shown to exist in ZFC. However, with additional axioms it is possible to prove the existence of such separable metric spaces.

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