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ON THE DENSITY OF SEQUENCE $\{n_k\xi\}$

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Introduction

In his paper *Problems and results in Diophantine approximations* II which appeared in [2] Erdös asked the following:

Given a sequence of integers $n_1 < n_2 < n_2 \cdots$ satisfying $n_{k+1}/n_k \geq \alpha > 1$, $k = 1, 2, \ldots$, is it true that there always exists an irrational ξ for which the sequence $\{n_k\xi\}$ is not everywhere dense?

Here $\{x\}$ denotes the fractional part of x.

Strzelecki [5] has shown that if $\alpha \geq (5)^{1/3}$, and (t_k) is a sequence of positive real numbers, not necessarily integers, with $t_{k+1}/t_k > \alpha$ then there is a ξ such that $\{t_k\xi\} \in [\beta, 1-\beta]$, $k = 1, 2, \ldots$, for some $\beta > 0$.

It is the purpose of this paper to provide a complete answer to the question of Erdös by providing the following.

THEOREM. *Let (t_n) be a sequence of positive numbers such that*

$$q_n = t_{n+1}/t_n \geq \alpha > 1 \quad \text{for} \quad n = 1, 2, \ldots \tag{1}$$

and let s_0 be a real number $0 < s_0 < 1$ then there exists a real number $\beta = \beta(\alpha, s_0) > 0$ and a set T of Hausdorff dimension at least s_0 such that if $\xi \in T$ then

$$\{t_k\xi\} \in [\beta, 1-\beta] \quad \text{for} \quad k = 1, 2, \ldots . \tag{2}$$

We have the following immediate corollary.

COROLLARY. *The set of numbers ξ such that $\{t_k\xi\}$ is not dense in the unit interval has Hausdorff dimension* 1.

A similar result has recently been obtained independently by B. de Mathan [3], [4].

Proof of the Theorem. We note that it is sufficient to prove the theorem under the additional restriction that $q_n \leq \alpha^2$, for we can form a new sequence (t'_n) from (t_n) by introducing new terms between t_k and t_{k+1} if $t_{k+1}/t_k > \alpha^2$, so that $\alpha \leq t'_{n+1}/t'_n \leq \alpha^2$, $n = 1, 2, \ldots$. Obviously if the assertion of the theorem holds for some sequence (t'_n) it holds for any sub-sequence (t_n) of (t'_n).

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