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2-COHOMOLOGY OF SOME UNITARY GROUPS

BY

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In [1], we showed that the 2-cohomology of the group $SU(n, q)$ with coefficients in the standard module $V = \mathbf{F}_{q^2}^n$ is generally zero. For $SU(2, q)$, which is, of course, equal to $SL(2, q^2)$, the only exceptions occur at $q = 2^k$ with $k \geq 2$; in unpublished work, McLaughlin has shown that the second cohomology group has dimension 1 over $\mathbf{F}_{q^2}$. For $n > 2$ and $q > 3$, the only possible exceptions are at $n = 3$ with $q = 4$ or 3^k and $n = 4$ with $q = 4$. In this paper, we prove that $H^2(SU(n, q), V)$ has dimension 1 over $\mathbf{F}_{q^2}$ in the first case and vanishes in the second. We also show that $H^2(SU(3, 3), V)$ is zero.

In Section 1, we outline some basic results on the cohomology of groups. In the second section, we compute $H^2(SU(3, q), V)$ with $q = 4$ or 3^k, $k > 1$, while the 2-cohomology of $SU(4, 4)$ is determined in the third section. Finally, we show $H^2(SU(3, 3), V) = 0$ in the fourth section.

1. In this section, we describe some results on the cohomology of groups which will be needed later. For a more complete discussion, the reader is referred to [2] and [5].

Let $1 \to A \to G \to X \to 1$ be an exact sequence of groups and let V be a (left) G-module. From the Lyndon-Hochschild-Serre spectral sequence we get the exact sequence

$$H^2(X, V^A) \to H^2(G, V)_0 \to H^1(X, H^1(A, V)),$$

where V^A denotes the set of A-fixed points of V and $H^2(G, V)_0$ is the kernel of the restriction map res: $H^2(G, V) - H^2(A, V)^X$.

If A and V are finite elementary abelian p-groups and $V^A = V$, we have the exact sequence of X-modules

$$0 \to \mathrm{Hom}\,(A, V) \xrightarrow{\mu} H^2(A, V) \xrightarrow{\varepsilon} \mathrm{Alt}_2\,(A, V) \to 0,$$

where $\mathrm{Alt}_2\,(A, V)$ is the group of alternating $\mathbf{F}_p$ bilinear forms from A to V. μ is the Bockstein operator with $\mu(h)$ equal to the class of the 2-cocycle

$$\mu_1(h)(a, b) = ((h(a) + h(b))^p - h(a)^p - h(b)^p)/p,$$

and ε is defined at the cocycle level by $\varepsilon(f)(a, b) = f(a, b) - f(b, a)$.

We shall be most interested in the case where A and V are vector spaces over some finite field K and $\dim_K A = 1$. In this situation we can take advantage of special direct sum decompositions of $\mathrm{Hom}\,(A, V)$ and $\mathrm{Alt}_2\,(A, V)$ to simplify

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