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ON ISOMETRIES OF THE BLOCH SPACE

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Introduction

Let Δ denote the open unit disc in the complex plane $\mathbf{C}$ and let Γ be the unit circle. The (normalized) set $\mathscr{B}$ of Bloch functions is defined as follows:

$$\mathscr{B} = \{f\colon f \text{ is holomorphic on } \Delta, f(0) = 0,$$
$$\text{and } \sup_{|z|<1} |f'(z)|(1 - |z|^2) \equiv M(f) < \infty\}.$$

With pointwise operations and $M(f) = \|f\|$, $\mathscr{B}$ becomes a nonseparable Banach space. Let $\mathscr{B}_0$ denote the closed subspace of $\mathscr{B}$ spanned by the polynomials. For many general properties of $\mathscr{B}$, see [1]. In [2], some function theoretic properties of the extreme points of the unit balls of $\mathscr{B}$ and $\mathscr{B}_0$ are investigated.

In this paper we characterize the (linear) isometries of $\mathscr{B}_0$ and the onto isometries of $\mathscr{B}$. Our description of these isometries closely parallels descriptions of isometries of many other spaces of analytic functions. See, for instance, [3], [4], [6], and [7]. Our work is patterned after a proof in [5, p. 141] of a theorem describing the isometries of function algebras. Thus as a first step we identify $\mathscr{B}$ with a subspace of $C_b(\Delta)$, the bounded continuous functions on Δ.

2. The Isometries of $\mathscr{B}_0$.

Let $C(\Delta)$ denote the continuous functions on Δ, and let

$$\mathscr{C} = \{f \in C(\Delta)\colon \|f\|_{\mathscr{C}} \equiv \sup_{|z|<1} |f(z)|(1 - |z|^2) < \infty\}.$$

Let $\mathscr{D} = \{f \in \mathscr{C}\colon f \text{ is holomorphic on } \Delta\}$. Then $\mathscr{D}$ is a closed subspace of $\mathscr{C}$ and the derivative mapping $D\colon f \to f'$ is a linear isometry of $\mathscr{B}$ onto $\mathscr{D}$. Let $\mathscr{D}_0 = D(\mathscr{B}_0)$.

Now define a mapping $\Phi\colon C_b(\Delta) \to \mathscr{C}$ by $(\Phi f)(z) = f(z)(1 - |z|^2)^{-1}$. Clearly Φ is an onto linear isometry. We denote Φ^{-1} by Ψ. Let $\Psi(\mathscr{D}) = \mathscr{A}$, $\Psi(\mathscr{D}_0) = \mathscr{A}_0$. We then have

$$\mathscr{A} = \{f'(z)(1 - |z|^2)\colon f \in \mathscr{B}\}, \qquad \mathscr{A}_0 = \{f'(z)(1 - |z|^2)\colon f \in \mathscr{B}_0\}$$

with $\mathscr{A}_0 \subset \mathscr{A} \subset C_b(\Delta)$ and $\mathscr{D}_0 \subset \mathscr{D} \subset C$. Indeed, $\mathscr{D}_0 \subset C_0(\Delta) \equiv$ all continuous functions on Δ which vanish on Γ.

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