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AN EXACT FORMULA FOR AN AVERAGE OF L-SERIES

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Let χ be a character of a prime p. As usual,

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \tau(\chi) = \sum_{r=1}^{p} \chi(r) e\left(\frac{r}{p}\right)$$

where s is a complex variable and $e(z) = e^{2\pi i z}$.

The average in question is

$$\sum_{\chi(-1)=-1} |L(1, \chi)|^2 = \frac{(p-1)^2(p-2)}{12p^2} \pi^2. \tag{1}$$

Chowla [1] gave the asymptotic value $(\pi^2 p)/12$ for the left hand side of (1) and noted that the asymptotic form of (1) was implicit in the work of Paley [2] and Selberg [3].

If $F(t)$ is the fractional part of t minus 1/2 and $e(t) = e^{2\pi i t}$, then

$$F(t) = -\frac{1}{2\pi i} \sum_n \frac{e(nt)}{n} \tag{2}$$

where n runs over all non zero integers. Let ψ be a non-principal character mod p. From (2),

$$\sum_{r=1}^{p-1} \bar{\psi}(r) F\left(\frac{r}{p}\right) = -\frac{1}{2\pi i} \sum_n \frac{1}{n} \sum_{r=1}^{p-1} \bar{\psi}(r) e\left(\frac{rn}{p}\right).$$

Since

$$\sum_{r=1}^{p-1} \bar{\psi}(r) e\left(\frac{rn}{p}\right) = \tau(\bar{\psi}) \psi(n)$$

we have

$$\sum_{r=1}^{p-1} \bar{\psi}(r) F\left(\frac{r}{p}\right) = -\frac{\tau(\bar{\psi})}{2\pi i} \begin{cases} 2L(1, \psi) & \text{if } \psi(-1) = -1 \\ 0 & \text{if } \psi(-1) = 1 \end{cases} \tag{3}$$

even when ψ is principal, as an easy computation shows. Thus,

$$\sum_{r,t=1}^{p-1} F\left(\frac{r}{p}\right) F\left(\frac{t}{p}\right) \bar{\psi}(r) \psi(t) = \begin{cases} \dfrac{p}{\pi^2} |L(1, \psi)|^2 & \text{if } \psi(-1) = -1 \\ 0 & \text{if } \psi(-1) = 1 \end{cases}$$

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