ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 2, Summer 1982

AN ALGEBRAIC CONCEPT OF SYMPLECTIC CURVATURE STRUCTURES

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1. Introduction

Given a pseudo-riemannian manifold $(\mathcal{M}, \sigma)$ Levi-Civitas unique torsionfree connection ∇ induces the canonical pseudo-riemannian curvature structure R by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for vector fields X, Y on $\mathcal{M}$. R is skew symmetric in X and Y, fulfills the first Bianchi identity (see (S.3) below) and defines a section $R(X, Y)$ of the pseudo-orthogonal Lie algebra bundle over $\mathcal{M}$. Singer and Thorpe [15] conversely defined a curvature structure on a pseudo-orthogonal vector space, here $(T_p\mathcal{M}, \sigma)$, as a (1, 3) tensor with these three axioms. The linear space spanned by these curvature structures can be taken as a typical fibre of a vector bundle over $\mathcal{M}$ in which the above canonical curvature structure is a section. The special types of pseudo-riemannian manifolds (of constant curvature, Einsteinian, etc.) usually are defined in terms of this section. Petrov first has given a basis in the space of curvature structures on a 4-dimensional Minkowski space.

In the following, on a symplectic vector space $(\mathbf{E}, \sigma)$ of finite dimension $2n$, an algebraic analog of such a pseudo-orthogonal curvature structure is developed, by changing the sign in the first axiom (S.1) and inserting the symplectic Lie algebra

$$\mathrm{sp}(\mathbf{E}, \sigma) = \{Q \text{ in end } \mathbf{E}/\sigma(Qx, y) + \sigma(x, Qy) = 0 \text{ for all } x, y \text{ in } \mathbf{E}\}$$

for the pseudo-orthogonal one in (S.3). In Sections 2 and 3 it is shown that most results on pseudo-orthogonal curvature structures can be overtaken almost literally. Especially Weyl's conformal curvature again defines a projector and hence a decomposition of the curvature space which is invariant under the induced action of the symplectic group $\mathrm{Sp}(\mathbf{E}, \sigma)$ on $(\mathbf{E}, \sigma)$ and its Lie algebra $\mathrm{sp}(\mathbf{E}, \sigma)$. Nomizu's characterization [14] of the kernel of this projector by the (Jordan algebra of) σ-selfadjoint endomorphisms on $\mathbf{E}$,

$$JA(\mathbf{E}, \sigma) = \{A \text{ in end } \mathbf{E}/\sigma(Ax, y) = \sigma(x, Ay) \text{ for all } x, y \text{ in } \mathbf{E}),$$

can be proved as well. The proofs are essentially those of riemannian differential geometry, as described for instance in [5], [6] and [8]. The following treatment is based on the work of Kulkarni [10], [11], Kowalski [9] and Nomizu

Received June 30, 1980.