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THE EXPONENTIAL FUNCTION CHARACTERIZED BY AN APPROXIMATE FUNCTIONAL EQUATION

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THEOREM. *Let p be real, $p > 1$. Let the complex-valued function $f(x)$ belong to the Lebesgue class $L^p(0, z)$ for each real $z > 0$, and satisfy*

$$(1) \qquad \lim_{z \to \infty} e^{-\varepsilon z} \int_0^z \int_0^z |f(x+y) - f(x)f(y)|^p \, dx \, dy = 0$$

for each fixed $\varepsilon > 0$.

Then either there is a (possibly complex) constant β so that

$$(2) \qquad f(x) = e^{\beta x}$$

almost surely for $x \geq 0$, or

$$(3) \qquad \lim_{z \to \infty} e^{-\varepsilon z} \int_0^z |f(x)|^p \, dx = 0$$

for each fixed $\varepsilon > 0$.

Each of the conditions (2), (3) *is sufficient to guarantee the validity of that at* (1).

The proof of this theorem depends upon the possibility of analytically continuing the solution of certain Riccati differential equations in the complex plane.

Since $||a| - |b|| \leq |a - b|$ condition (1) is also satisfied by $|f(x)|$. *For the time being we shall assume that $f(x)$ is real and non-negative.*

It is convenient to define

$$J(z) = \int_0^z f(x) \, dx$$

for $z \geq 0$.

LEMMA 1. *There is a non-negative constant B so that*

$$J(z) \leq e^{Bz}$$

holds for $z \geq 3$.

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