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AN INEQUALITY BETWEEN THE VOLUME AND THE CONVEXITY RADIUS OF A RIEMANNIAN MANIFOLD

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1. In [2], Marcel Berger is interested in finding lower bounds on the volume $v(g)$ of a compact n-dimensional Riemannian manifold (M, g) in terms of the injectivity radius $i(g)$ and convexity radius $c(g)$. Recently, Berger [3] proved that

$$v(g) \geq (\alpha(n)/\pi^n)i^n(g)$$

and consequently that

$$v(g) \geq (\alpha(n)/(\pi/2)^n)c^n(g)$$

with equality holding if and only if (M, g) is a sphere of constant curvature. (Here $\alpha(n)$ is the volume of the unit n-sphere.) It is reasonable to expect that stronger inequalities hold when M is not homeomorphic to a sphere.

In this paper we exhibit a constant $\mu'(3) > \alpha(3)/(\pi/2)^3$ such that for any non-simply-connected 3-dimensional manifold (M, g), $v(g) \geq \mu'(3)c^3(g)$. Along the way we refine Loewner's theorem [8], [1] thereby giving a lower bound on the area of a torus or Klein bottle in terms of the two shortest nonhomotopically trivial closed curves.

Throughout this paper, let ρ be the distance function associated to the Riemannian manifold (M, g). The open geodesic ball of radius $r > 0$ centered at the point $p \in M$ is defined by

$$B(p; r) = \{q \in M: \rho(p, q) < r\}.$$

Thus, if $0 < r \leq i(g)$, then the exponential map is a diffeomorphism of the open ball of radius r centered at the origin in the tangent space at p onto $B(p; r)$. Also, if $0 < r \leq c(g)$, then $B(p; r)$ is strongly convex, meaning that any two points of $B(p; r)$ are connected by a unique minimizing geodesic and this geodesic lies in $B(p; r)$. We will frequently use the relation $2c(g) \leq i(g)$ (Remark 1.6 of [2]).

2. Let (M, g) be a compact non-simply-connected Riemannian manifold. Let $l(g)$ denote the length of the shortest nonhomotopically-trivial closed curve. Clearly, $l(g) \geq 2i(g) \geq 4c(g)$. Let γ be a shortest non-

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