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APPROXIMATION BY POLYNOMIALS OF GIVEN LENGTH

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I. Introduction

The number of non-zero coefficients of a polynomial P is denoted by $l(P)$ and called the length of P. Accordingly, expressions of the form

$$\sum_{1 \leqslant k \leqslant m} c_k x^{n_k} \quad \text{and} \quad \sum_{1 \leqslant k \leqslant m} \gamma_k e^{is_k t},$$

where n_k, s_k are integers, $n_k \geqslant 0$, represent algebraic, resp. trigonometric polynomials of length $\leqslant m$.

We shall consider algebraic polynomials of length $\leqslant m$ on some interval $[a, b]$; by a change of variable $x = cx'$, $c \neq 0$, we may assume that the interval is of the form $[\delta, 1]$, where $-1 \leqslant \delta < 1$. (We cannot reduce all considerations to just one interval, like $[0, 1]$, since the length of an algebraic polynomial is not invariant under translation.)

The length of a polynomial is as natural a concept as the degree; however, the length is much less convenient to work with: polynomials of length $\leqslant m$ do not form a vector-space, nor even a convex set; in the standard functional spaces the set of polynomials of length $\leqslant m$ and norm less than or equal to 1 is not a compact set; polynomial leg-ngth, as we mentioned, is not invariant under translation; etc. There seems to be only one well-known positive statement on algebraic polynomials of length $\leqslant m$: they have at most $m - 1$ zeroes on $(0, + \infty)$.

To illustrate the curious consequences of that lack of previously established results, consider the statement: the polynomials of length $\leqslant m$ form a closed subset in $C[a, b]$. That is, of course, true. But even that quite simple statement can not be considered trivial, because in order to prove it, one needs to establish first some other, more basic result, like the estimate in the lemma below.

The main result of this paper is the existence of best approximation by polynomials of length $\leqslant m$ in $C[a, b]$, and, more generally, in $L^p[a, b]$, $1 \leqslant p \leqslant + \infty$ (Theorem 2).

The proof of Theorem 2 is based on (i) a generalization of the familiar fact that if V is a finite-dimensional subspace of the normed vector space B, then every element in B has a best approximation in V (that generalization

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