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FOLIATIONS WITH LOCALLY REDUCTIVE NORMAL BUNDLE

BY

ROBERT A. BLUMENTHAL

1. Introduction

Let M be a connected smooth manifold and let $\mathscr{F}$ be a smooth codimension q foliation of M. Let $T(M)$ be the tangent bundle of M and let $E \subset T(M)$ be the subbundle consisting of vectors tangent to the leaves of $\mathscr{F}$. Let $Q = T(M)/E$ be the normal bundle of $\mathscr{F}$ and let $\pi: T(M) \to Q$ be the natural projection. We shall denote by $\chi(M)$, $\Gamma(E)$, and $\Gamma(Q)$ the spaces of smooth sections of the vector bundles $T(M)$, E, and Q respectively. Let

$$\nabla: \chi(M) \times \Gamma(Q) \to \Gamma(Q)$$

be a connection on Q. Following [10] we say that ∇ is an adapted connection if $\nabla_X Y = \pi([X, \tilde{Y}])$ for all $X \in \Gamma(E)$ and all $Y \in \Gamma(Q)$ where $\tilde{Y} \in \chi(M)$ is any vector field satisfying $\pi(\tilde{Y}) = Y$. Such a connection is called basic in [3] and is characterized by the condition that the parallel translation which it induces along a curve lying in a leaf of $\mathscr{F}$ coincides with the natural parallel translation along the leaves. Let $T: \chi(M) \times \chi(M) \to \Gamma(Q)$ be the torsion of ∇, that is, $T(X, Y) = \nabla_X(\pi Y) - \nabla_Y(\pi X) - \pi([X, Y])$. Then ∇ is adapted if and only if $i(X)T = 0$ for all $X \in \Gamma(E)$ where $i(X)T$ denotes the one-form on M with values in Q given by $(i(X)T)(Y) = T(X, Y)$ for $Y \in \chi(M)$. Let

$$R: \chi(M) \times \chi(M) \to \mathrm{Hom}_{\mathbf{R}}\,(\Gamma(Q), \Gamma(Q))$$

be the curvature of ∇, that is, $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for $X, Y \in \chi(M)$, $Z \in \Gamma(Q)$. Following [10] we say that the adapted connection ∇ is basic if $i(X)R = 0$ for all $X \in \Gamma(E)$ where $i(X)R$ denotes the one-form on M with values in the bundle $\mathrm{End}\,(Q)$ given by $(i(X)R)(Y) = R(X, Y)$ for $Y \in \chi(M)$.

In Section 2 we study complete basic connections and prove:

THEOREM 1. *Let M and N be connected manifolds and let $f: M \to N$ be a submersion. Let ∇ be a connection on $Q = T(M)/\ker(f_*)$ and $\bar{\nabla}$ a linear connection on N such that $f^{-1}(\bar{\nabla}) = \nabla$. If ∇ is complete, then $f: M \to N$ is a locally trivial fiber bundle and $\bar{\nabla}$ is also complete.*

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