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LINEAR TOPOLOGICAL PROPERTIES OF THE HARMONIC HARDY SPACES h^p FOR $0 < p < 1$

BY

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Dedicated to the memory of David L. Williams

1. Introduction

Let Δ denote the open unit disc of the complex plane. In this paper we study, for $0 < p < 1$, the space h^p of complex valued functions u harmonic on Δ, for which

$$\|u\|_p^p = \sup_{0 \le r < 1} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta/2\pi < \infty. \tag{1}$$

If $p \geq 1$ then the functional $\|\cdot\|_p$ is a norm which makes h^p into a Banach space. But if $0 < p < 1$ it is instead the *p-norm* $\|\cdot\|_p^p$ which is subadditive, and used to induce the translation-invariant metric. In either case metric convergence implies uniform convergence on compact subsets of Δ, so even if $0 < p < 1$, the space h^p is complete, has enough continuous linear functionals to separate points; and its topology is "natural" for harmonic functions.

For $p \geq 1$ the h^p spaces are well known objects with many desirable properties. For example [5; Chapters 2, 3, and 4]:

(i) The Poisson integral establishes an isometric isomorphism between h^p and a classical Banach space: $L^p(\partial\Delta)$ if $p > 1$, and the space of complex Borel measures on $\partial\Delta$ if $p = 1$.

(ii) Each function in h^1 has a finite non-tangential limit at almost every point of $\partial\Delta$.

(iii) The conjugate function operator $u \to \tilde{u}$ is well behaved. If $1 < p < \infty$, the M. Riesz theorem asserts that h^p is "self-conjugate", that is, if u is in h^p, then so is its harmonic conjugate $\tilde{u}$. This is not true for h^1, but here Kolmogorov's theorem provides a substitute: if $u \in h^1$ then $\tilde{u} \in h^p$ for all $p < 1$.

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