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HIRONAKA GROUP SCHEMES

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We will be working in the category of k-schemes where k is a field. A Hironaka group scheme is a closed subgroup scheme H of an affine space $\mathbf{A}^n$ which is invariant under scalar multiplication by $\mathbf{G}_m$; i.e., H is a cone. Hironaka [1] introduced such group schemes as the stabilizer in $\mathbf{A}^n$ of a closed cone in $\mathbf{A}^n$. As a Hironaka group scheme is its own stabilizer any Hironaka group scheme arises this way.

We intend to classify roughly all Hironaka group schemes and explain their structure. Our presentation avoids Dieudonné modules and Hopf algebras. If $\mathrm{char}(k) = 0$, a Hironaka group scheme is just a vector subspace. Hence in this paper we will assume that $\mathrm{char}(k)$ is a prime p.

Let $H \subset \mathbf{A}^n$ be a Hironaka group scheme. Then the quotient $\mathbf{A}^n/H$ is an algebraic group whose formation commutes with base extension. Furthermore we have an induced action of $\mathbf{G}_m$ on $\mathbf{A}^n/H$. Here $\mathbf{G}_m$ acts via automorphisms of this quotient group. The central result is:

THEOREM. *$\mathbf{A}^n/H$ is $\mathbf{G}_m$-equivariantly isomorphic to a finite sum $\oplus\, \mathbf{A}^{m_q}(q)$ where the q's and m_*'s are positive integers and $\mathbf{A}^{m_q}(q)$ is the affine space $\mathbf{A}^{m_q}$ with the $\mathbf{G}_m$-action given by $t^*x = t^q x$.*

Proof. Let B be the ring of regular functions on $\mathbf{A}^n/H$ and let m^* be the comultiplication. Let P be the k-subspace of primitive element in B; i.e.,

$$P = \{ f \in B \mid m^* f = f \otimes 1 + 1 \otimes f \}.$$

By linear algebra the formation of P commutes with base extension and P is contained in the maximal ideal m of functions vanishing at the identity of $\mathbf{A}^n/H$.

LEMMA. *The induced k-linear mapping $d\colon P \to m/m^2$ is surjective.*

Proof. By base extension it is enough to check this when k is algebraically closed. Now $\mathbf{A}^n/H$ is an algebraic group variety which is commutative and its

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