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# KNESER COLORINGS OF POLYHEDRA

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## 1. Introduction

(1.1) It is well known that if a 1-dimensional simplicial complex, i.e., a "graph", $K^1$, embeds in a 2-dimensional manifold $M^2$, then its chromatic number is less than a certain constant $c$, which depends only on the topology of $M^2$. We have proved elsewhere various generalisations of this result which apply to higher dimensional simplicial complexes $K^n$: see [18], [19] and [20].

In this paper we turn things around and show that if a simplicial complex $K^n$ can be suitably colored by not too many colors, then it p.l. embeds in a given $\mathbf{R}^m$. As typical specimens of such results we have the following two:

THEOREM 2 (2.5.1). *Let $G(K^n_\square)$ denote the graph whose vertices are pairs $(v,\theta)$ where $v$ is a vertex of $K^n$ and $\theta$ a maximal simplex of $K^n$ not containing $v$, with $(v_1, \theta_1)$ adjacent to $(v_2, \theta_2)$ iff $v_1 \in \theta_2$ and $v_2 \in \theta_1$. If $G(K^n_\square)$ has chromatic number $\leq m + 1$ and $2m \geq 3\,(n + 1)$, or else $n = 1$ and $m = 2$, then $K^n$ p.l. embeds in $\mathbf{R}^m$.*

THEOREM 6 (3.2.1). *Let $G(X^n)$ denote the graph whose vertices $X^n_i$ are closures of the non-singular edge-less components of the underlying polyhedron $X^n$ of $K^n$, with $X^n_i$ adjacent to $X^n_j$ iff $X^n_i$ is disjoint from $X^n_j$. If $G(X^n)$ is bichromatic and $n \neq 2$ then $K^n$ p.l. embeds in $\mathbf{R}^{2n}$.*

Note that Theorem 6 above includes the well known fact that an $n$-pseudo-manifold p.l. embeds in $\mathbf{R}^{2n}$. The hypotheses of this theorem are relaxed considerably in Theorem 8 (3.4.2) whose statement involves some equivariant cohomology.

In Theorem 2 above, $K^n_\square$ denotes a self-dual poset, *the dual deleted product*, which we associate canonically to each simplicial complex $K^n$. Theorems 3 and 4 of (2.5) are analogues of Theorem 2 for graphs $G(K^n_\Sigma)$ arising out of some sub self-dual posets $K^n_\Sigma$ of $K^n_\square$. Theorem 3 is in fact a common generalization of Theorem 2 above and the Lovász-Kneser Theorem [12] which appears in this setting only as a very special colorability implies embeddability

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