ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 1990

SUBNORMAL AND ASCENDANT SUBGROUPS WITH RANK RESTRICTIONS

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Let $\mathfrak{F}_r^*$ denote the class of groups with finite abelian section rank (thus $H \in \mathfrak{F}_r^*$ if and only if every elementary abelian p-section of H is finite for all primes p). If H and K are subnormal $\mathfrak{F}_r^*$-subgroups of the group G, then the subgroup $J = \langle H, K \rangle$ is not necessarily subnormal in G (even if G belongs to $\mathfrak{F}_r^*$). This is shown in [4, §5], using a construction due to Zassenhaus and Hall. However, it is proved in the same paper that if $G \in \mathfrak{F}_r^*$ then J is ascendant in G. Here we make the following improvement on that result.

THEOREM. *Let $H_1, H_2, \ldots, H_n$ be subnormal $\mathfrak{F}_r^*$-subgroups of the group G, and let $J = \langle H_1, H_2, \ldots, H_n \rangle$. Then*
(a) *J belongs to $\mathfrak{F}_r^*$,*
(b) *J is ascendant in G.*

The number of subgroups H must be finite for either of these conclusions to hold, as may be seen by considering an infinite elementary abelian p-group in the case of (a), and a group of type $G = C_p \operatorname{wr} C_{p\infty}$ in the case of (b). (In the latter, the self-normalizing "top group" is a union of cyclic p-groups, each of which is subnormal in G.) Further, if we weaken the hypothesis of the theorem by replacing "subnormal" by "ascendant", then the conclusion (a) does not follow.

Example. Let p be a fixed prime. For each $i = 1, 2, \ldots,$ let H_i be a group with presentation

$$\langle h_{i,i}, h_{i,2}, \ldots : (h_{i,1})^p = 1, (h_{i,j+1})^p = h_{i,j} \ (j = 1, 2, \ldots) \rangle.$$

Then each H_i is clearly a group of type $C_{p\infty}$ and thus of rank 1. Let H denote the direct product of the H_i, $i = 1, 2, \ldots,$ and let x be the automorphism of H defined by

$$x\colon (h_{i,j}) \to (h_{i,j})(h_{i+1,j-1}); \qquad i, j = 1, 2, \ldots$$

(where, for each i, $(h_{i+1,0})$ is interpreted as the identity element). Note that x

Received January 8, 1988.