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MINIMIZING NORMS OF POLYNOMIALS UNDER CONSTRAINTS ON THE DISTRIBUTION OF THE ROOTS

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Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be an univariate polynomial with complex coefficients; we note $\|P\|_\infty = \max_{|z|=1} |P(z)|$. A well-known result of Erdös-Turan (see [6]) asserts that if the ratio

$$\frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}}$$

is not too large, then the roots are uniformly distributed in different angles with vertex at the origin. To be precise, for every α, β with $0 \leq \alpha \leq \beta \leq 2\pi$, if $N_{\alpha,\beta}$ is the number of roots z_j with $\arg z_j \in [\alpha, \beta]$, then

$$\left| N_{\alpha,\beta} - \frac{\beta - \alpha}{2\pi} n \right| \leq c \sqrt{n \log \frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}}}$$

where n is the degree of P.

Erdös-Turan obtained $c = 16$, a value which was improved later by Ganelius (see [7]): $c \sim 2.619$. Taking the polynomial $(z-1)^n$, we see that $c \geq 1/\sqrt{\log 2} \sim 1.201$.

The result of Erdös-Turan concerns the distribution of roots in the whole plane, but it does not yield optimal estimates if specific constraints are laid upon the roots, especially if they are required to lie in a half-plane, or a sector. However, the polynomials whose roots lie in the open half-plane $\{\operatorname{Re} z < 0\}$ play a particularly prominent role in physics; they are called stable polynomials (see [8]). By extension, we call "stable" any polynomial all of whose roots lie in the closed half-plane $\{\operatorname{Re} z \leq 0\}$.

In the present paper, we consider the following problem: let P be a polynomial whose roots lie in a sector $\{|\operatorname{Arg} z| \geq \theta \geq \pi/2\}$; what distribution of the roots minimizes the quantity $\|P\|_\infty / \sqrt{|a_0 a_n|}$?

Beside the norm $\|\cdot\|_\infty$, we consider $\|P\|_2 = (\sum_{j=0}^{n} |a_j|^2)^{1/2}$. These two norms have very different implications: the first one, for a signal, controls

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