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PARAMETRIZING THE SOLUTIONS OF AN ANALYTIC DIFFERENTIAL EQUATION

LOU VAN DEN DRIES[1]

Dedicated to the memory of Lee A. Rubel (1928–1995)

Introduction

One of Rubel's many research problems about algebraic differential equations (cf. [5, problem 21] and [6, problem 28]) is as follows:

Given a sequence (z_n) *of distinct complex numbers tending to infinity, and any sequence* (w_n) *of complex numbers, does there exist a differentially algebraic entire function* f *such that* $f(z_n) = w_n$ *for all* $n \in \mathbf{N}$?

It is classical that the answer is "yes" if the "differentially algebraic" requirement is dropped. Below, in (2.4), we show the answer is "no" in our case: *Given any such sequence* (z_n), *the set of sequences* $(w_n) \in \mathbf{C}^{\mathbf{N}}$ *for which there is a differentially algebraic entire function* f *with* $f(z_n) = w_n$ *for all* n, *is meagre in the sequence space* $\mathbf{C}^{\mathbf{N}}$ *equipped with the product topology.* (Rubel had already shown in [6] that if one prescribes not only the values of $f(z_n)$ but also those of $f^{(j)}(z_n)$ for $1 \leq j \leq n$, then the interpolation problem is in general not solvable.)

The goal of this paper is to reproduce Malgrange's local parametrization of the solutions of an analytic differential equation, which is perhaps not as widely known as it should be, and to indicate its role in answering questions of this sort.

Let us now define some of our terms precisely. By *region in* $\mathbf{C}$ we mean a nonempty connected open subset of the complex plane $\mathbf{C}$, and a holomorphic function f on such a region is said to be *differentially algebraic* (or DA, for short) if there is a nonzero polynomial $P(Z, W, W', \dots, W^{(m)})$ in the variables $Z, W, W', \dots, W^{(m)}$ over $\mathbf{C}$ such that $P(z, f(z), f'(z), \dots, f^{(m)}(z)) = 0$ for all z in the region.

We then also say that f is a *solution* of the algebraic differential equation (ADE)

$$P(z, w, w', \dots, w^{(m)}) = 0.$$

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