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ON THE PARITY OF PARTITION FUNCTIONS

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Section 1

For $n = 1, 2, \ldots, p(n)$ denotes the number of unrestricted partitions of n, and $q(n)$ the number of partitions of n into distinct parts, and we write $p(0) = q(0) = 1,\ p(-1) = q(-1) = p(-2) = q(-2) = \cdots = 0$. If N, b are positive integers, a is an integer, then let $E_{a,b}(N)$ denote the number of non-negative integers n such that $n \leq N$ and $p(n) \equiv a \bmod b$.

Starting out from a question of Ramanujan, in 1920 MacMahon [3] gave an algorithm for determining the parity of $p(n)$. Since then, many papers have been written on the parity of $p(n)$. In particular, in 1959 Kolberg proved that $p(n)$ assumes both even and odd values infinitely often (for $n \geq 0$). His proof was based on Euler's identity

$$p(n) + \sum_{k \geq 1} (-1)^k (p(n - s_k) + p(n - t_k)) = 0$$

where $s_k = \frac{1}{2}k(3k - 1)$, $t_k = \frac{1}{2}k(3k + 1)$ and the summation extends over all terms with a non-negative argument. It follows from this identity that

$$p(n) + \sum_{k \geq 1} (p(n - s_k) + p(n - t_k)) \equiv 0 \bmod 2. \tag{1}$$

Other proofs have been given for Kolberg's theorem by Newman [5] and Fabrykowski and Subbarao [1]. Parkin and Shanks [7] have computed the parity of $p(n)$ up to $n = 2039999$. Their calculation suggest that $E_{0,2}(N) \sim E_{1,2}(N) \sim N/2$. Mirsky [4] has proved the only quantitative result on the frequency of the odd values and even values of $p(n)$. In fact, starting out

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