ILLINOIS JOURNAL OF MATHEMATICS
Volume 42, Number 4, Winter 1998

SQUARES OF CHARACTERS THAT ARE THE SUM OF ALL IRREDUCIBLE CHARACTERS

S. M. GAGOLA, JR. AND M. L. LEWIS

1. Introduction

We study here the structure of groups G which possess an irreducible character χ with the property that χ^2 is the sum of all the irreducible characters of G. (All groups considered here are finite, and by character we mean complex character, that is, the character afforded by a representation over the field of complex numbers.)

Previous to our present work, E. Abboud in [1] showed that G is a split extension of an elementary abelian 2-group by an elementary abelian 3-group when G' is assumed abelian. We are able to prove here:

(1.1) THEOREM. *If G is a finite solvable group for which there exists an irreducible character χ such that $\chi^2 = \sum_{\psi \in \mathrm{Irr}(G)} \psi$, then G is an internal direct product of copies of the symmetric group S_3.*

Certainly, Theorem (1.1) suggests that the hypotheses are fairly restrictive, at least for solvable groups. Other examples of this situation (already noted in [1]) are the groups $G = \mathrm{SL}_2(2^n)$ for all $n \geq 1$ where χ is the Steinberg character of degree 2^n. Notice that the symmetric group S_3 occurs as the first term of this family, but the remaining members are all simple groups. It is easy to check that direct products of examples produce further examples. (Conversely, direct factors of examples also serve as examples.) In view of these examples, it seems natural to generalize Theorem (1.1) to $\mathcal{S}$-groups: groups all of whose nonsolvable composition factors are isomorphic to members of the collection $\mathcal{S} = \{\mathrm{SL}_2(2^n) \mid n \geq 2\}$. We obtain:

(1.2) THEOREM. *Let G be a finite group for which there exists an irreducible character χ such that $\chi^2 = \sum_{\psi \in \mathrm{Irr}(G)} \psi$. If G is an $\mathcal{S}$-group, then G is an internal direct product $G = X_1 \dot{\times} \cdots \dot{\times} X_k$ of groups X_i that are isomorphic to groups in the family $\{S_3\} \cup \mathcal{S}$.*

Notice that Theorem (1.1) is an immediate corollary of Theorem (1.2), as S_3 is the only solvable member of the family $\{S_3\} \cup \mathcal{S}$.

Received August 25, 1997.
1991 Mathematics Subject Classification. Primary 20C15.