AN EXTENSION OF RAUCH'S METRIC COMPARISON THEOREM AND SOME APPLICATIONS[1]

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1. Introduction

In [8] Toponogov proved a theorem relating the angles of a triangle in a Riemannian manifold V to those of a triangle having the same lengths of sides in the simply connected two-dimensional space which has constant curvature equal to the lower bound of sectional curvatures of V. Toponogov's proof used a theorem of Alexandrov for surfaces. But for triangles whose side-lengths are not too big in comparison to the upper bound of sectional curvatures of V, Toponogov's theorem is equivalent to Rauch's metric comparison theorem [6, p. 36]. In this article we want to give a new proof of Toponogov's theorem, a proof using only Rauch's metric comparison theorem. Strictly speaking the proof will use too a slight extension of Rauch's theorem; this extension will be proved in §2 as Theorem 1. In itself, this extension is of interest; we give in §3 a first application of it as Proposition 1. In §4 another application of the extension is a very short proof of a theorem of Toponogov concerning manifolds of maximum diameter: Theorem 2 below. And in §5 we give the new proof of Toponogov's theorem.

2. The extension

Definitions and notations are those of [1], [2], [3]. Moreover by $S_n(\delta)$ we shall denote the simply connected n-dimensional manifold whose curvature is constant and equal to δ (and $S_2(\delta) = S(\delta)$); that is, if $\delta > 0$, a sphere; if $\delta = 0$, a euclidean space; if $\delta < 0$, a hyperbolic space. In this paper V will always be a complete Riemannian manifold of dimension n whose sectional curvatures form a set $\mathrm{curv}(V)$ satisfying $\delta \leqq \mathrm{curv}(V) \leqq 1$. Rauch's metric comparison theorem works with a one-parameter family of geodesics of V issuing from a fixed point $p \in V$ and asserts (if some nonconjugacy hypothesis is verified) that the length of the curve of V built up by the extremities of the geodesics of the family is less than or equal to the length of the curve built up by the extremities of the one-parameter family of geodesics in $S_n(\delta)$ associated in a natural way with the starting family in V. Now it can be helpful to have an analogous theorem in which the family of geodesics one works with is formed by geodesics whose starting points run through a given geodesic, and which are orthogonal at these points to the given geodesic. We shall now write down in a more precise way the material for the theorem we anticipate.

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