ON THE DECOMPOSITION THEORY FOR KRULL VALUATIONS

BY

P. RIBENBOIM

Let K be a field endowed with a Krull valuation v, $L \mid K$ a finite Galoisian extension, $\mathcal{V} = \{w = w_1, w_2, \cdots, w_g\}$ the set of distinct prolongations of v to L. We define and study the decomposition field and decomposition group associated with a *distinguished set* $\mathcal{E}$ of valuations, $\mathcal{E} \subseteq \mathcal{V}$.

Among other results, we obtain a new proof that the value group $w(Z)$ and the residue-class field Z/w of the decomposition field Z of w in $L \mid K$ are respectively the same as those of the ground field K: $w(Z) = v(K)$, $Z/w = K/v$; cf. [1], [4, pp. 70 ff.].

Finally, the theory is applied to define the decomposition field of a prolongation of the valuation v to a finite extension of K, which may be neither normal nor separable.

An example is given to show that the results indicated cannot be improved.

1. Known results and a technical lemma

Let w_1, w_2 be valuations of a field L, and x_1, x_2 nonzero elements of L. We say that the pair (w_1, x_1) is *compatible* with the pair (w_2, x_2) in case

$$(w_1 \wedge w_2)(x_1) = (w_1 \wedge w_2)(x_2),$$

where $w_1 \wedge w_2$ denotes the greatest lower bound of w_1, w_2 in the ordered set of valuations of L (cf. [4, p. 43] or [3]).

This relation is transitive: If (w_1, x_1) is compatible with (w_2, x_2), and if (w_2, x_2) is compatible with (w_3, x_3), let us consider $w_1 \wedge w_2$ and $w_2 \wedge w_3$. Since both valuations are coarser than w_2, one is coarser than the other, say $w_1 \wedge w_2 \geqq w_2 \wedge w_3$; hence $w_1 \wedge w_3 = w_2 \wedge w_3$. Thus, if either $(w_1 \wedge w_2)(y) = 0$ or $(w_2 \wedge w_3)(y) = 0$, we have $(w_1 \wedge w_3)(y) = 0$. This implies that

$$(w_1 \wedge w_3)(x_1/x_3) = (w_1 \wedge w_3)(x_1/x_2) + (w_1 \wedge w_3)(x_2/x_3) = 0,$$

showing that (w_1, x_1) is compatible with (w_3, x_3).

More generally, the set $\{(w_1, x_1), (w_2, x_2), \cdots, (w_g, x_g)\}$ is said to be *compatible* when (w_i, x_i) is compatible with (w_j, x_j), for any $i \neq j$.

The following theorems will be used (cf. [3]):

APPROXIMATION THEOREM. *If $w_1, \cdots, w_g$ are pairwise incomparable valuations of L, if $x_1, \cdots, x_g \in L$ are such that*

$$\{(w_1, x_1), (w_2, x_2), \cdots, (w_g, x_g)\}$$

Received July 18, 1961; received in revised form January 4, 1962.