PERIODIC HOMEOMORPHISMS OF THE 3-SPHERE[1]

BY

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1. Statement of results

Let $\mathfrak{M}$ be a triangulated 3-sphere, and let f be a periodic simplicial homeomorphism of $\mathfrak{M}$ onto itself. Suppose that f preserves orientation and has a fixed point; let F be the fixed-point set of f; and let n be the period of f. It has been shown by P. A. Smith [S][2] that when n is a prime, F is always a (simple closed) polygon; and we shall show, in the last section of the present paper, that for arbitrary n the same conclusion follows. In the rest of this paper, therefore, we shall assume that F is a polygon. A well-known conjecture due to Smith, discussed by Eilenberg in [E], asserts that F is never knotted.

A partial solution of Smith's problem has been given by Montgomery and Samelson [MS]. They have shown that if f is an involution (i.e., is of period 2), then (1) if F is a simplicial standard torus knot, then F is unknotted, and (2) if F is unknotted, then f is equivalent to a rotation.

In the present paper, we generalize the second of these results, to homeomorphisms of arbitrary period. Thus our main result is:

1.1. THEOREM. *If f: $\mathfrak{M} \to \mathfrak{M}$ is periodic and preserves orientation, and F is unknotted, then f is equivalent to a rotation.*

The proof is based on the following preliminary result:

1.2. THEOREM. *There is a polyhedral disk with handles M_1 such that the boundary of M_1 is F and such that the iterated images*

$$M_i = f^{i-1}(M_1)$$

intersect one another only in F.

Here by a disk with handles we mean, of course, a compact, connected, orientable 2-manifold with boundary, bounded by a 1-sphere.

Theorem 1.2 has been proved, for involutions, by Montgomery and Samelson.

2. 2-spines of 3-dimensional complexes

Let $\mathfrak{K}$ be a complex, and let n be a positive integer. Then $\beta_n \mathfrak{K}$ denotes the set of all points of $\mathfrak{K}$ that do not have open neighborhoods in $\mathfrak{K}$, homeomorphic to Euclidean n-space E^n. The "n-dimensional interior" $\mathfrak{K} - \beta_n \mathfrak{K}$

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[2] Letters in square brackets refer to the bibliography at the end of the paper.