TWO THEOREMS ON AUTOMORPHIC FUNCTIONS

Dedicated to Hans Rademacher on his seventieth birthday

BY
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1. Recently Rademacher gave a new proof of the fundamental theorem that a modular function belonging to a modular congruence subgroup and which is regular and bounded in the upper half-plane is a constant [6]. His argument relies on the divergence of the Poincaré series

$$\sum_{V} | cz + d |^{-2}, \qquad V = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where V runs over the principal congruence subgroup of level N with the restriction $-\frac{1}{2}N \leqq \Re(V(i)) < \frac{1}{2}N$.

The natural generalization of the modular group is the class of horocyclic groups (Grenzkreisgruppen, Fuchsian groups of the first kind) that have fundamental regions with a finite number of sides. *We call this class* $\mathfrak{F}$. In attempting to apply Rademacher's reasoning one must first prove the divergence of the analogous Poincaré series, a fact we state as Theorem 1. For convenience we shall assume our groups are defined on the unit disk $\mathfrak{U}$. As is well known, every linear transformation mapping $\mathfrak{U}$ on itself can be written as

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}, \qquad a\bar{a} - c\bar{c} = 1.$$

THEOREM 1. *If* $\Gamma \in \mathfrak{F}$, *then*

$$\sum_{n=0}^{\infty} | c_n z + \bar{a}_n |^{-2} = \infty \tag{1}$$

for each $z \in \mathfrak{U}$. *Here*

$$\left\{ V_n = \begin{pmatrix} a_n & \bar{c}_n \\ c_n & \bar{a}_n \end{pmatrix}, n \geqq 0; V_0 = I \right\}$$

is an enumeration of the elements of Γ.

Theorem 1 is classical ([3], pp. 255–258). The first object of this paper is to present a new proof. After this it will be easy to extend Rademacher's argument and so obtain

THEOREM 2. *A function regular and bounded in* $\mathfrak{U}$ *and automorphic on a group* $\Gamma \in \mathfrak{F}$ *is a constant.*

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