AN INEQUALITY SUGGESTED BY THE THEORY OF STATISTICAL INFERENCE

Dedicated to Hans Rademacher on the occasion of his seventieth birthday

BY

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We consider Borel regions W in the n-dimensional Euclidean space R_n and two functions F_1, F_2 such that $\int_W dF_1$, $\int_W dF_2$ are nonnegative. The integral $\int_W dF_1 = P_1(W)$ is called the size of W. The integral $\int_W dF_2 = P_2(W)$ will be called the power of W.

A family L of regions in R_n will be called an additive family if sums, intersections, and differences of L are again in L.

In the following definitions and theorems all regions considered will be regions of an additive family L. To avoid cumbersome language we shall however just speak of regions. A most powerful region W will mean a region of the family L such that $P_2(W) \geqq P_2(W')$ for all $W' \in L$ for which $P_1(W) = P_1(W')$. We shall also assume that our regions satisfy the following condition.

(i) *If W is any region of size α, and if $0 \leqq \beta < \alpha$, then W has a subregion of size β.*

Condition (i) obviously implies

(i′) *If W is any region of size α, and if $\alpha = \sum_{i=1}^{n} \beta_i$, then $W = \sum W_i$, where W_i has size $\beta_i \geqq 0$.*

LEMMA 1. *Let W be a most powerful region of size α, and W_1 any subregion of W of size $\beta \leqq \alpha$. Let K be any region of size β and $K \cap W$ empty. Then $P_2(K) \leqq P_2(W_1)$.*

Otherwise the region $W - W_1 + K$ would have size α and higher power than W.

LEMMA 2. *Let W be a most powerful region of size α, and W^* any subregion of W of size $\beta \leqq \alpha$. Let K be any region of size $k\beta$ such that $K \cap W$ is empty; then*

$$kP_2(W^*) \geqq P_2(K). \tag{1}$$

Proof. If $P_2(K) = \infty$, then K must have a subregion K_1 of size $\leqq \beta$ such that $P_2(K_1) = \infty$. By Lemma 1 this implies $P_2(W^*) = \infty$, and (1) holds.

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