EXTENSIONS OF SHEAVES OF ALGEBRAS

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Introduction

It is well known that, since sheaves of modules over a fixed sheaf of rings R on a topological space X form an abelian category, the set of equivalence classes of extensions of a sheaf A'' by a sheaf A', i.e., of exact sequences

$$0 \to A' \to A \to A'' \to 0,$$

is in 1–1 correspondence with the first derived functor $\mathrm{Ext}_R^1(A'', A')$ of $\mathrm{Hom}_R(A'', A')$. This statement about sheaves corresponds exactly to the analogous statement about modules. Similarly, there is a classification theorem for extensions of an associative algebra Λ by a kernel A with trivial multiplication which asserts that the set $F(\Lambda, A)$ of equivalence classes of such extensions is in 1–1 correspondence with $\mathrm{Ext}^2_{\Lambda^e}(\Lambda, A)$, where Λ^e denotes the enveloping algebra of Λ. It is natural to ask whether or not the same result holds for sheaves of associative algebras. It will be shown that, in general, this is not the case, but that under appropriate hypotheses there is an exact sequence

$$\to \check{H}^1(X, A) \to F(\Lambda, A) \to \mathrm{Ext}^2_{\Lambda^e}(\Lambda, A) \to \check{H}^2(X, A) \to \cdots$$

where the symbols refer to sheaves of algebras and where $\check{H}^*(X, A)$ is the Čech cohomology of X with coefficients in A.

The paper is divided into three parts. In the first section it is shown that the groups $\mathrm{Ext}^n_R(B, A)$ can be calculated from a "weakly projective and coherent" resolution of B instead of an injective resolution of A. The main result of the second section is that if Λ is itself a weakly projective and coherent sheaf of associative algebras, then the usual standard complex is a suitable resolution of Λ. Similar results are given for sheaves of supplemented algebras and for sheaves of Lie algebras. In the third section it is shown that the corresponding extensions can be classified by the cohomology groups of a subcomplex of the bicomplex of Čech cochains of X with coefficients in the appropriate standard complexes. This leads to the indicated exact sequence.

It is assumed throughout the paper that X is paracompact Hausdorff, and essential use is made of the uniqueness theorem for cohomology with coefficients in a differential sheaf. As this theorem does not seem to be readily available, a short proof of it which was suggested by J. C. Moore is given in the first section. Also, Proposition 1.1 was included in a series of his lectures.

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