ALGEBRAIC LIE ALGEBRAS AND REPRESENTATIVE FUNCTIONS SUPPLEMENT

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The purpose of this note is to supplement the results obtained in [2] concerning the structure of the algebra of the representative functions on the universal enveloping algebra of a Lie algebra. Theorems 1, 2, and 3 are analogues of results obtained jointly with G. D. Mostow for Lie groups (to appear elsewhere), and the proofs are based on the same ideas, although the technicalities involved are rather different.

In order not to disrupt the continuity later, we begin with a simple fact concerning the universal enveloping algebra of a nilpotent Lie algebra.

LEMMA 1. *Let P be a nilpotent Lie algebra of finite dimension over an arbitrary field F, and choose a basis $x_1, \cdots, x_n$ for P such that each commutator $[x_i, x_j]$ is an F-linear combination of x_k's with $k < \min(i, j)$. Let U denote the universal enveloping algebra of P, and let $U^{[q]}$ stand for the subspace of U that is spanned by the ordered monomials $x_n^{e_n} \cdots x_1^{e_1}$ with $e_1 + \cdots + e_n \geqq q$. Then, for each q, there is an exponent q_1, such that $P^{q_1}U \subset U^{[q]}$.*

Proof. We define a weight function w on the set of the ordered monomials in our basis elements such that $w(1) = 0$, $w(x_i) = 2^{n-i}$, and the weight of an ordered monomial is the sum of the weights of its factors. Every element $u \in U$ can be written uniquely as an F-linear combination of ordered monomials. For $u \neq 0$, we define $w(u)$ to be the minimum taken by w on the set of ordered monomials occurring with a nonzero coefficient in the standard expression for u. Then we shall have $u \in U^{[q]}$ whenever $w(u) \geqq 2^{n-1}q$. Now we claim that, if $u \neq 0$, $w(x_i u) > w(u)$. Evidently, it suffices to establish this in the case where u is an ordered monomial. In that case, one easily shows by induction on the degree of u that $w(x_i u) \geqq w(x_i) + w(u)$. This establishes our claim, and the conclusion of Lemma 1 follows immediately.

Now we must recall some of the notation and results of [2, Section 6]. Let L be a finite-dimensional Lie algebra over a field F of characteristic 0. Let A denote the radical of L, and set $T = [L, A]$. Let S be a maximal semisimple subalgebra of L. $\mathbf{R}(L)$ denotes the algebra of the representative functions on the universal enveloping algebra $U(L)$ of L. We have defined a subalgebra $\mathbf{R}^S(L)$ of $\mathbf{R}(L)$ (as the canonical image of $\mathbf{R}(L/A)$ in $\mathbf{R}(L)$), such that the restriction to $U(S)$ maps $\mathbf{R}^S(L)$ isomorphically onto $\mathbf{R}(S)$, and a subalgebra $\mathbf{R}^A(L)$, such that the restriction to $U(A)$ maps $\mathbf{R}^A(L)$ isomorphically onto the restriction image $\mathbf{R}(L)_A$ of $\mathbf{R}(L)$ in $\mathbf{R}(A)$. Then

$$\mathbf{R}(L) = \mathbf{R}^S(L)\mathbf{R}^A(L) \approx \mathbf{R}^S(L) \otimes \mathbf{R}^A(L).$$

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