A CLASS OF TOEPLITZ FORMS AND THEIR APPLICATION TO PROBABILITY THEORY[1]

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The subject matter of this paper belongs both to analysis and to probability theory as our results were suggested by, and form a natural continuation of, well known problems in both fields. First we extend the theory of the asymptotic behavior of Toeplitz forms, this part of the work being based on the recent book of U. Grenander and G. Szegö [4]. While our asymptotic results are of a more refined character than theirs, this is achieved at the expense of generality. We consider only a very special class of Toeplitz forms—just those, in fact which, as we shall explain, correspond in a natural way to the class of symmetric lattice random variables. Thanks to this correspondence we then obtain new results (and a new approach to old results) concerning the limiting behavior of functionals of the partial sums of symmetric lattice random variables.

The datum of our problem will be a "probability sequence" $\{c_k\}$ satisfying the conditions

$$\begin{gathered} c_k = c_{-k} \geqq 0 \qquad \text{for } k = 0, \pm 1, \pm 2, \cdots, \\ \sum_{-\infty}^{\infty} c_k = 1, \qquad 0 < \sum_{-\infty}^{\infty} k^2 c_k = \sigma^2 < \infty, \\ \text{g.c.d. } [k \mid k > 0,\, c_k > 0] = 1. \end{gathered} \tag{0.1}$$

In terms of the sequence $\{c_k\}$ we define the sequence of Toeplitz matrices

$$(T_{ij}(N)) = T(N) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_N \\ c_1 & c_0 & c_1 & \cdots & c_{N-1} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdots & \cdot\cdot \\ c_N & c_{N-1} & c_{N-2} & \cdots & c_0 \end{bmatrix}, \qquad N \geqq 0. \tag{0.2}$$

$T^k(N)$ will denote the k^{th} power of $T(N)$ for $k \geqq 1$; $T^0(N) = I(N) = I$, the identity matrix with $N + 1$ rows and columns. Finally, we shall denote by $H(N)$ the inverse of the matrix $I - T(N)$.

For $|z| = 1$, let

$$\phi(z) = \sum_{-\infty}^{\infty} c_k z^k. \tag{0.3}$$

It is clear from (0.1) that $1 - \phi(z) \geqq 0$ for $|z| = 1$, and it is one of the basic facts of the theory of Toeplitz forms [4] that there exists a unique sequence of polynomials which satisfy the following conditions:

(i) for every integer $n \geqq 0$, $p_n(z)$ is a polynomial of degree n in which

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