INVERSION OF TOEPLITZ MATRICES II[1]

BY

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1. Introduction

With a function $\varphi(\theta) \in L_1(0, 2\pi)$, $\varphi(\theta) \sim \sum_{-\infty}^{\infty} c_k e^{ik\theta}$, is associated the *semi-infinite Toeplitz matrix* $T_\varphi = (c_{j-k})_{0 \leqq j,k < \infty}$. In case $\sum | c_k | < \infty$, T_φ represents a bounded operator on the space l_∞^+ of bounded sequences

$$X = \{x_0, x_1, \cdots\},$$

and in [1] a necessary and sufficient condition was found for the invertibility of T_φ (i.e., the existence of a bounded inverse for T_φ), namely that $\varphi(\theta) \neq 0$ and $\Delta_{-\pi \leqq \theta \leqq \pi} \arg \varphi(\theta) = 0$. If $\varphi(\theta) \in L_\infty$, T_φ represents a bounded operator on the space l_2^+ of square-summable sequences, and in §3 of [1] sufficient conditions were obtained for invertibility in this situation.

The purpose of the present paper is to obtain conditions which are necessary as well as sufficient for invertibility of T_φ as an operator on l_2^+. That the situation is quite different in the l_∞^+ and l_2^+ cases can be seen, for instance, from the fact that in the former, the set of φ for which T_φ is invertible forms a group, while in the latter we may have T_φ invertible but T_{φ^2} not (Corollary 2 of Theorem IV).

As in all problems of Wiener-Hopf type, and this is one, the basic idea is a certain type of factorization. In our case, the idea is that of writing T_φ as the product of triangular Toeplitz matrices (which amounts to a factorization of φ), the question of invertibility for these being simpler since any two triangular Toeplitz matrices of the same type commute. Thus, roughly speaking, if φ is sufficiently nice, we can factor T_φ and then invert each factor, thus obtaining the inverse of T_φ. This gives rise to sufficient conditions for invertibility, as in [1, §3]. Now in the l_∞^+ theory it turned out that the φ's for which this could be carried out were *exactly* those giving rise to invertible Toeplitz matrices; thus the invertibility of T_φ implies the existence of a suitable factorization of φ. It is the content of Theorem I of the present paper that this situation prevails also in the l_2^+ case. From this result we easily settle the invertibility question for triangular and self-adjoint Toeplitz matrices.

For general Toeplitz matrices we have been unable to find a simple criterion for invertibility; there is one however (Theorem IV) in case $\arg \varphi(\theta)$ is reasonably well-behaved.

Before proceeding, we introduce some notation. For $f(\theta) \in L_p(0, 2\pi)$,

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