BOOLEAN RINGS AND BANACH LATTICES[1]

BY

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1. Introduction

Let X be a Banach lattice of measurable functions. If $\chi_e \in X$ is the characteristic function of a set e, $\Phi(e) = \| \chi_e \|$ is a function defined on a certain Boolean ring of sets. In this paper we consider the following problem. If a function $\Phi(e)$ is given on a Boolean ring B, what are the conditions under which B can be imbedded into a vector lattice X and Φ extended into a norm on X? Under what conditions on Φ is it possible to postulate some additional properties of X? Answers to such questions are given in Sections 2, 3, 5. This leads in Section 6 to a natural generalization of certain spaces introduced by one of the authors [4] under the name of spaces Λ. We consider abstract Boolean rings B and correspondingly functions in the sense of Carathéodory [3]. The reader may substitute for this, if he so wishes, Boolean rings of sets and point-functions. This substitution would not lead to any simplification of the proofs.

2. Extension of a multiply subadditive function into a norm

Let B be a Boolean ring, i.e., a distributive, relatively complemented lattice with zero element (a Boolean ring is a Boolean algebra if and only if it contains a unit). Let $\Phi(e)$ be a real valued function defined on B. We will discuss extensions of B into a vector lattice S such that unions of disjoint elements of B become sums, intersections become products, and the order is preserved, and at the same time extensions of Φ into a seminorm on S.

The smallest extension of B of this kind is the vector lattice S of step-functions. The elements of S are formal sums $x = \sum_{k=1}^{n} a_k e_k$ (where e_k is also the characteristic function of the set e_k) with an obvious identification rule (see [5], [3]).

A seminorm $P(x)$ on a vector space satisfies the following relations:

$$\text{(a)} \quad P(x) \geqq 0, \qquad \text{(b)} \quad P(ax) = | a | P(x),$$

$$\text{(c)} \quad P(x_1 + x_2) \leqq P(x_1) + P(x_2).$$

Other natural conditions for $P(x)$ are

$$\text{(d)} \quad P(x) \leqq P(y) \quad \text{for } 0 \leqq x \leqq y, \qquad \text{(e)} \quad P(| x |) = P(x).$$

Theorem 1. (α). *A real valued function Φ on B has an extension P onto S which is a seminorm (we call such Φ norm-generating) if and only if Φ satisfies*

$$\text{(i)} \qquad \Phi(e) \leqq \sum_{k=1}^{n} | a_k | \Phi(e_k) \qquad \textit{for } e = \sum a_k e_k, \quad e, e_k \in B;$$

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