ON THE PURITY OF BRANCH LOCI IN REGULAR LOCAL RINGS[1]

BY
MASAYOSHI NAGATA

The purity of branch loci, proved by Zariski [5], is as follows:

Let V be an algebraic variety of a function field K over a ground field k, and let L be a finite separable algebraic extension of K. Then the branch locus of V with respect to the derived normal variety $N(V; L)$ of V in L is purely divisorial locally at every simple branch point.

The purpose of the present paper is to prove the generalization of the above result to general regular local rings, which can be stated as follows:

THEOREM. *Let P be a regular local ring, and let Q be a normal local ring which dominates P and which is a ring of quotients of a finite separable integral extension of P. If, for every prime ideal $\mathfrak{p}$ of rank 1 in P, $\mathfrak{p}$ is unramified in Q, then Q is unramified over P.*

Here, unramifiedness should be understood as follows:[2]

A quasi-local ring[3] Q' dominating another quasi-local ring Q'' is said to be unramified over Q'' if (i) the maximal ideal of Q' is generated by that of Q'', and (ii) the residue class field of Q' is separable over that of Q''. A prime ideal $\mathfrak{p}'$ in Q' is said to be unramified over Q'' if $Q'_{\mathfrak{p}'}$ is unramified over $Q''_{(\mathfrak{p}' \cap Q'')}$. A prime ideal $\mathfrak{p}''$ in Q'' is said to be unramified in Q' if every prime ideal of Q' lying over $\mathfrak{p}''$ is unramified over Q''.

We say that a ring R is of finite type over another ring S if R is a ring of quotients of a ring R' which is a finite module over S.

In §1, we shall prove a criterion of unramifiedness. In §2, we shall reduce the theorem to the case where P is complete, and in §3, we shall prove the theorem by induction on rank P, while the case where rank $P = 2$ is assumed to be known, because the case was proved by Serre and Auslander-Buchsbaum independently, and, though Serre is not publishing his proof, Auslander and Buchsbaum are publishing their proof.

1. A criterion of unramified extensions

We shall recall a well known easy lemma, which we shall call Krull-Azumaya's lemma:[4]

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[2] There are some other notions of "unramifiedness".

[3] A ring (commutative and having the identity) is said to be quasi-local if it has only one maximal ideal.

[4] Once the writer called this "Azumaya's lemma". But since this is an immediate corollary of Krull's lemma (which is the case where $N = 0$), and since Azumaya formulated this lemma and called attention to its convenience in the first instance, the writer now wants to call this "Krull-Azumaya's lemma".