A VARIATIONAL METHOD FOR TRIGONOMETRIC POLYNOMIALS[1]

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1. Introduction

Let $f(x)$ be a trigonometric polynomial. We consider a linear functional $\mathfrak{L}$ defined by

$$\mathfrak{L}(f) = \sum_{\nu=1}^{m} \sum_{j=0}^{n\nu} \alpha_\nu^{(j)} f^{(j)}(x_\nu),$$

where x_ν, α_ν are given real numbers, $0 \leqq x_\nu < 2\pi$, with the x_ν all different. We suppose that $\alpha_\nu^{(n_\nu)} \neq 0$ and that $n_\nu > 0$ for at least one ν. We call

$$l = n_1 + \cdots + n_m + m$$

the order of $\mathfrak{L}$; thus $f'(a) - f'(-a)$ is a functional of order 4. We are interested in the maximum of $| \mathfrak{L}(f) |$ when f runs through the class of trigonometric polynomials of type n which satisfy $|f(x)| \leqq 1$ for real x. (It is convenient to say that a trigonometric polynomial is of type n if it is of degree at most n; a trigonometric polynomial of type n is an entire function of exponential type n.) In looking for this maximum it is enough to consider the subclass $\mathfrak{J}_n$ whose members are in addition real for real x. For, if θ is real, we have $e^{i\theta}f(z) = f_1(z) + f_2(z)$, where f_1 and f_2 are elements of $\mathfrak{J}_n$. Since $\mathfrak{L}(e^{i\theta}f) = e^{i\theta}\mathfrak{L}(f)$, we can choose θ so that $\mathfrak{L}(e^{i\theta}f) = | \mathfrak{L}(f) |$, and so $\mathfrak{L}(f_1) = | \mathfrak{L}(f) |$. Hence the maximum of $| \mathfrak{L}(f) |$ is attained, if at all, for an f in $\mathfrak{J}_n$, and indeed for one for which $\mathfrak{L}(f) > 0$.

When $\mathfrak{L}(f) = f'(a)$, we have S. Bernstein's theorem that $|f'(a)| \leqq n$ when $|f'(x)| \leqq 1$ for all x. Here the bound for $| \mathfrak{L}(f) |$ is the same no matter which point a is selected; this is no longer true in the general case.

Bernstein's theorem on trigonometric polynomials is a special case of his theorem on entire functions of exponential type: if $f(z)$ is an entire function of exponential type τ (which we may suppose is real for real x), and $|f(x)| \leqq 1$ for all real x, then $|f'(x)| \leqq \tau$ for all real x. This does not happen for more general functionals $\mathfrak{L}$. In fact, Schaeffer and I [1] found that the maximum of $| \mathfrak{L}(f) |$ in this class $\mathfrak{F}_\tau$ of entire functions is not, in general, attained for a trigonometric polynomial f. However, methods similar to those used in [1] still work for the class $\mathfrak{J}_n$. The general result is stated in §3 below; in §4 it is applied to the special functional $\lambda n^2 f(0) + f''(0)$. As corollaries, we obtain two theorems for ordinary polynomials. Further applications will be given elsewhere.

The problem of maximizing the functional $f'(a) - f'(-a)$ is equivalent to the problem of maximizing $p_n'(x)$ for a given x on $(-1, 1)$ when the poly-

Received October 11, 1957; received in revised form December 3, 1957.

[1] Research supported by the National Science Foundation.