AN AXIOMATIZATION OF THE HOMOTOPY GROUPS

BY

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1. Introduction

In the present paper an axiomatic characterization of homotopy groups will be given. The possibility of such a characterization was conjectured by S. Eilenberg and N. E. Steenrod in [1]. Another such axiomatization, which is essentially different from the present one, is due to J-P. Serre, J. W. Milnor [6], M. Kuranishi [5] and S. T. Hu [2]. The main difference is that we do *not* postulate that "π_0 is the set of the components". Nonessential is the fact that we consider only absolute groups. The results will be stated in terms of c.s.s. complexes. Free use will be made of the definitions and results of [3], [4], and [7].

In an appendix we discuss the influence of the first two axioms (homotopy and exactness) on $\pi_1(K(Z, 1))$, which plays a role similar to that of the coefficient group in homology theory.

2. The main result

Let $\mathcal{S}$ be the category of c.s.s. complexes with base point ([3], §2), and let $\mathcal{S}_c$ be the subcategory of the c.s.s. complexes which are of the weak homotopy type of a countable c.s.s. complex ([4], §6). We shall define the notion of a theory of homotopy groups on a subcategory $\mathcal{S}' \subset \mathcal{S}$ and state uniqueness theorems for theories of homotopy groups on the categories $\mathcal{S}_c$ and $\mathcal{S}$.

Let $\mathcal{G}$ be the category of groups, and let $\mathcal{G}_c$ be the subcategory of the countable groups. All groups will be written multiplicatively. A group consisting of one element will be denoted by 1.

DEFINITION 1. A *theory of homotopy groups* $\{\pi_i, \partial_i\}$ on a subcategory $\mathcal{S}' \subset \mathcal{S}$ is a collection which contains for every integer $n > 0$

(a) a functor $\pi_n : \mathcal{S}' \to \mathcal{G}$,

(b) a function ∂_{n+1} which assigns to every fibre sequence $F \xrightarrow{q} E \xrightarrow{p} B$ in $\mathcal{S}'$ ([3], §3) a homomorphism $\partial_{n+1}(q, p) : \pi_{n+1}(B) \to \pi_n(F)$ satisfying the naturality condition:

If commutativity holds in the diagram

$$\begin{tikzcd} F \arrow[r, "q"] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "e"] & B \arrow[d, "b"] \\ F' \arrow[r, "q'"] & E' \arrow[r, "p'"] & B' \end{tikzcd}$$

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