ON SUBDETERMINANTS OF DOUBLY STOCHASTIC MATRICES[1]

BY MARVIN MARCUS

In this note we obtain an inequality for the euclidean norm of an n-square complex matrix $A = (A_{ij})$, (Theorem 1). This is used to give lower bounds for the rank of A and in particular for the rank of a doubly stochastic matrix. We then distinguish (Theorems 2 and 3) a certain simple set of matrices among all doubly stochastic matrices in terms of possible values for the subdeterminants. In particular a characterization of the permutation matrices as a subclass of doubly stochastic matrices is given in terms of bounds on the subdeterminants.

We proceed to describe some notation to be used throughout. A typical r-square subdeterminant of A will be denoted by $d_r(A)$, det A will be the determinant of A. The sum over all $\binom{n}{r}^2$ choices of some function φ of the d_r will be denoted by

$$\sum \varphi(d_r(A)),$$

and the norm of A is given by

$$\| A \|^2 = \sum \mid d_1(A) \mid^2 = \sum_{i,j} \mid A_{ij} \mid^2.$$

The i^{th} row vector of A is $A_{(i)}$, and the j^{th} column vector is $A^{(j)}$. The rank of A is $\rho(A)$; I_k is the k-square identity matrix; 0_k is the k-square matrix of zeros; $A \dotplus B$ is the direct sum of A and B; and the conjugate transpose of A is A^*. A doubly stochastic (d.s.) matrix A is one which satisfies

$$\sum_{j=1}^{n} A_{ij} = 1, \qquad i = 1, \cdots, n$$

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$$A_{ij} \geqq 0, \qquad i, j = 1, \cdots, n.$$

The r^{th} symmetric function of the letters $a_1, \cdots, a_k$ is $E_r(a_1, \cdots, a_k)$.

In [4] H. Richter proved for an arbitrary n-square complex matrix A that[2]

$$\text{(1)} \qquad \| (\det A)\, A^{-1} \|^2 \leqq n^{-(n-2)} \| A \|^{2(n-1)}$$

with equality if and only if AA^* is a scalar matrix.

The first result is an extension of (1).

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[2] The same result with a simpler proof appeared recently in a note of L. MIRSKY (Arch. Math., vol. 7 (1956), p. 276).