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FINITELY GENERATED FUNCTION FIELDS AND COMPLEXITY IN POTENTIAL THEORY IN THE PLANE

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1. Introduction. On a simply connected domain in the plane, the Riemann mapping function can be used to pull back the classical kernel functions of potential theory from the unit disc, and it follows that the kernel functions are given as simple rational combinations of two functions of one complex variable, the Riemann map, and its derivative. Indeed, if a is a point in a simply connected domain $\Omega \neq \mathbb{C}$ and $f_a(z)$ is the Riemann mapping function mapping Ω one-to-one onto the unit disc with $f_a(a) = 0$ and $f_a'(a) > 0$, then the Bergman kernel $K(z,w)$ associated to Ω is given by

$$K(z,w) = \frac{f_a'(z)\overline{f_a'(w)}}{\pi\big(1 - f_a(z)\overline{f_a(w)}\big)^2}.$$

Another way to write the same formula is

$$K(z,w) = \frac{cK(z,a)\overline{K(w,a)}}{\big(1 - f_a(z)\overline{f_a(w)}\big)^2},$$

where $c = \pi/f_a'(a)^2$. In this paper, I shall prove that, similarly, on an n-connected domain Ω such that no boundary component is a point, the Bergman kernel associated to Ω is a rational combination of only *three* basic functions of *one* complex variable. To be precise, there exists a formula of the form

$$K(z,w) = R\big(f_a(z), f_a'(z), K(z,b), \overline{f_a(w)}, \overline{f_a'(w)}, \overline{K(w,b)}\big), \tag{1.1}$$

where R is a *rational* function on $\mathbb{C}^6$, f_a is an Ahlfors map (which is a branched n-to-one covering map of Ω onto the unit disc with $f_a(a) = 0$ and $f_a'(a) > 0$ and maximal), and a and b are fixed points in Ω. I also prove that there is an irreducible polynomial of two complex variables $P(z,w)$ such that $P(f_a(z), K(z,b)/f_a'(z)) \equiv 0$, and so it follows that $K(z,b)$ is an algebraic function of f_a and f_a'. Consequently, formula (1.1) yields that $K(z,w)$ is an algebraic function of $f_a(z)$, $f_a'(z)$, and the conjugates of $f_a(w)$ and $f_a'(w)$. The polynomial $P(z,w)$ is a rather interesting algebraic geometric object attached to Ω. We show that it is conformally invariant, but we do not explore its possibly deeper significance here.

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