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PAIR CORRELATION OF FOUR-DIMENSIONAL FLAT TORI

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1. Introduction. Berry and Tabor conjecture in [1] that the distribution of local spacings between the eigenvalues of the quantization of the Hamiltonian for generic completely integrable systems follows the distribution of local spacings between random numbers. (This is often called the "Poissonian" distribution.) In [5], Sarnak shows that the pair correlations of eigenvalues on almost all two-dimensional flat tori have this behavior. The key is that for two-dimensional flat tori, the Berry-Tabor conjecture reduces to a statement about the values at integers of homogeneous two-variable quadratic forms (see also [2] for another approach to the spacing of these forms). In [6] the author extends Sarnak's results to almost all nth-degree forms in n variables in a measure-theoretic sense. In this paper we consider a set of measure zero (and thus not covered by [6]) among all four-variable homogeneous quartic forms: those that are the squares of four-variable homogeneous quadratics. These correspond to the eigenvalues of four-dimensional flat tori (and thus the Berry-Tabor conjecture) in a manner that we now explain.

Consider a positive-definite homogeneous polynomial $P(x_1,\dots,x_4)$ of degree 4. We are interested in the values taken by P at integers, a discrete set that can be ordered

$$0=\lambda_0<\lambda_1\le\lambda_2\le\cdots.$$

(We make the "desymmetrized" restriction that we do not count $P(\vec{x})$ and $P(-\vec{x})$ separately, but as only one instance.) The number of λ's less than N (counted with multiplicity) is essentially $c_P N$, where c_P is the volume of the region $\{P(\vec{x})<1\}/(\vec{x}\sim-\vec{x})$. The average spacing between λ's is thus constant, so we may investigate the distribution of the local spacings $\lambda_j-\lambda_k$ and the consecutive spacings $\lambda_{j+1}-\lambda_j$. We define

$$R_P(a,b,N)=\frac{|\{\lambda_j,\lambda_k<N \mid j\neq k, a\le\lambda_j-\lambda_k\le b\}|}{N} \tag{1}$$

to be their pair correlations and

$$E_P(a,b,N)=\frac{|\{\lambda_j\le N \mid a\le\lambda_{j+1}-\lambda_j\le b\}|}{N} \tag{2}$$

to be their consecutive spacing distribution. If the λ's are distributed evenly by a random process, then with probability one,

$$\lim_{T\to\infty}R_P(a,b,N)=c_P^2(b-a)$$

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