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ROTATION NUMBERS OF HAMILTONIAN ISOTOPIES IN COMPLEX PROJECTIVE SPACES

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In this paper, we prove the existence of symplectic invariants—we call them *rotation numbers*—for the time-1 map of a Hamiltonian isotopy in $\mathbb{C}\mathrm{P}^{n-1}$. These invariants bear interesting properties related to the geometry of the fixed-point set of the symplectic map. They are obtained by lifting the problem to the linear space $\mathbb{C}^n$, and then using invariant *generating functions* to define them as solutions of a certain finite-dimensional min-max method.

1. Statement of the result. The space $\mathbb{C}^n$ ($n \geqslant 2$) is endowed with its standard Euclidean structure, denoted by $\langle \cdot, \cdot \rangle$. We take its standard symplectic form Ω to be $\Omega(z, z') = \langle iz, z' \rangle$ for all z, z' in $\mathbb{C}^n$.

Let S^{2n-1} be the unit sphere in $\mathbb{C}^n$; the group S^1 acts on S^{2n-1}, with quotient space $S^{2n-1}/S^1 = \mathbb{C}\mathrm{P}^{n-1}$. We denote by $\pi : S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ the projection and by $i : S^{2n-1} \to \mathbb{C}^n$ the inclusion. There is a unique symplectic structure ω on $\mathbb{C}\mathrm{P}^{n-1}$ that satisfies the relation $i^*\Omega = \pi^*\omega$ (see [12]). An easy computation shows that, for the case $n = 2$, the symplectic area of $\mathbb{C}P^1$ equals π.

Let $h = (h_t)_{t \in [0,1]}$ be a time-dependent Hamiltonian on $\mathbb{C}\mathrm{P}^{n-1}$, and $(\varphi_t)_{t \in [0,1]}$ be its associated Hamiltonian isotopy. Recall that $(\varphi_t)_{t \in [0,1]}$ is obtained by integrating the vector field $(X_t)_{t \in [0,1]}$ defined by $dh_t = i_{X_t}\omega$ for t in $[0, 1]$.

A number, the *action* of x, is associated in a standard way to any fixed point x of φ_1 as follows. Consider the contractible loop $\gamma = (t \mapsto \varphi_t(x), t \in [0, 1])$ in $\mathbb{C}\mathrm{P}^{n-1}$, and choose any 2-disc D such that $\partial D = \gamma$. Then the symplectic area of D is a real number defined modulo π, and is denoted by $a(x)$. The action $c(x)$ of x is

$$c(x) := -\frac{1}{\pi}\left(a(x) + \int_0^1 h_t(\varphi_t(x))\, dt\right) \in \mathbb{R}/\mathbb{Z}.$$

Our result is now the following theorem.

THEOREM 1.1. *There exist n rotation numbers* $0 < t_1 \leqslant \cdots \leqslant t_n \leqslant 1$ *with the following properties.*

(a) *The image in* $S^1 \cong \mathbb{R}/\mathbb{Z}$ *of the sequence* $(t_1, \ldots, t_n)$ *depends only on the isotopy* $(\varphi_t)_{t \in [0,1]}$, *up to a cyclic shift of the indices and to a global rotation of* S^1. *It also depends continuously (for the* C^1*-topology) on the isotopy.*

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