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AN IMPROVED UPPER BOUND FOR THE 3-DIMENSIONAL DIMER PROBLEM

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1. Introduction. The dimer problem in dimension 3 is one of the classical unsolved problems in solid-state chemistry. It is as follows: Define a *brick* to be a d-dimensional ($d \geqslant 2$) rectangular parallelepiped with sides of integer lengths; we always assume its volume to be even. A brick of volume 2 is called a *dimer*. The problem is to determine the number $f(a_1, \dots, a_d)$ of dimer tilings of the brick with sides of lengths $a_1, \dots, a_d$.

It is remarkable that for $d = 2$ this number can be expressed in closed form, as was shown independently in [6] by Kasteleyn and in [14] by Temperley and Fisher. In contrast, for $d \geqslant 3$ no such formulas are known. Moreover, the number of dimer tilings of a brick of dimension at least 3 is not known even asymptotically. More precisely, Hammersley [4] proved that the sequence $1/(a_1 \cdots a_d) \log f(a_1, \dots, a_d)$ approaches a finite limit ℓ_d as $a_i \to \infty$, $i = 1, \dots, d$. However, the exact value of ℓ_d is not known for $d \geqslant 3$. The most important case for applications is the case $d = 3$ (see [5] and [7]).

Various upper and lower bounds for ℓ_3 have been proved as far back as six decades ago, when Fowler and Rushbrooke [3] showed (assuming, based on physical and heuristic arguments, that the limit defining ℓ_d exists) that

$$0 \leqslant \ell_3 \leqslant \frac{1}{2}\log 3 = 0.54931\dots.$$

This upper bound was improved by Minc [7] to $1/12 \log 6! = 0.54827\dots$, which represents the best upper bound previously known.

On the other hand, since ℓ_d is clearly a nondecreasing function of d, we obtain that a lower bound for ℓ_3 is given by ℓ_2, which was determined by Kasteleyn [6] and Temperley and Fisher [14] to be

$$\ell_2 = 1/\pi \sum_{r \geqslant 0} (-1)^r/(2r+1)^2 = 0.29156\dots.$$

This lower bound has been improved several times, first by Fisher (see [1]) to 0.30187, then by Hammersley [5] to 0.418347, and finally by Priezzhev [11] to 0.419989. A conjecture due to Schrijver and Valiant [12] on lower bounds for

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