

DISCRETE CONVOLUTION-REARRANGEMENT INEQUALITIES AND THE FABER-KRAHN INEQUALITY ON REGULAR TREES

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1. Introduction. Recall the classical Riesz convolution-rearrangement inequality in [24] (see [9] for a more general result)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y) \, dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^{\#}(x)g^{\#}(x-y)h^{\#}(y) \, dx dy$$

for positive functions on \mathbb{R} , where, given a function F on \mathbb{R} , the function $F^{\#}$ is the *symmetric decreasing rearrangement* of F . That is, the unique lower-semicontinuous function on \mathbb{R} , which is equimeasurable with F , is (not necessarily strictly) decreasing on $[0, \infty)$, and satisfies $F^{\#}(x) = F^{\#}(-x)$ for all x . Of special interest is the case where g is already symmetric decreasing, that is, where $g^{\#} = g$. That case can be expressed as

$$(1.1) \quad \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)K(|x-y|)h(y) \, dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^{\#}(x)K(|x-y|)h^{\#}(y) \, dx dy$$

for a decreasing function K on $[0, \infty)$. Inequalities like (1.1) are valid for a number of different types of rearrangements on \mathbb{R}^n , \mathbb{S}^n , and \mathbb{H}^n and often have inter-

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