Vol. 91, No. 2 DUKE MATHEMATICAL JOURNAL © 1998

ON TORSION LINE BUNDLES AND TORSION POINTS ON ABELIAN VARIETIES

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In this paper, we study torsion line bundles on abelian schemes and their restriction to the subschemes of torsion points. As we explain below, we are motivated by problems on the Galois module structure of rings of integers obtained by adjoining values of abelian functions at torsion points.

By definition, an abelian scheme $A \xrightarrow{f} S$ of relative dimension g is a group scheme that is proper and smooth with geometrically connected fibers of dimension g. Denote by $0 : S \to A$ the identity section. For any line bundle $\mathscr{L}$ on A, a rigidification of $\mathscr{L}$ is an isomorphism $\mathscr{O}_S \xrightarrow{\sim} 0^*\mathscr{L}$. For each nonzero integer m, we can consider the (closed) subgroup scheme $A_m \to S$ of m-torsion points of A. The group scheme A_m is finite and locally free of rank m^{2g} over S.

The first aim of this paper is to show the following theorem.

THEOREM A. *Assume $g = 1$ and $\gcd(6, m) = 1$, and let $\mathscr{L}$ be a torsion line bundle on A with a rigidification. Then the restriction of $\mathscr{L}$ to the subscheme of m-torsion points A_m is a trivial line bundle.*

We can phrase this more symmetrically, as follows.

THEOREM B. *Assume $g = 1$, and denote by $\mathscr{P}$ the Poincaré line bundle on $A \times_S A$. Assume that one of the two nonzero integers n and m is coprime to 6. Then the restriction of $\mathscr{P}$ to the subscheme $A_n \times_S A_m$ is a trivial line bundle.*

The proofs actually give trivializations for these restrictions. When $g \geqslant 2$, the situation is dramatically different. We will show the following.

THEOREM C. *Assume that $g \geqslant 2$. For any two distinct prime numbers r, l, there is a smooth affine curve D over a finite field of characteristic r, an abelian scheme $A \to D$ of relative dimension g, and an l-torsion line bundle $\mathscr{L}$ on A with a rigidification, such that the restriction of $\mathscr{L}$ to A_l is not trivial.*

The restriction of (torsion) line bundles on the subschemes of torsion points of elliptic curves has been studied extensively in a different formulation as a problem on the Galois module structure of the rings of integers in certain fields that are obtained by adjoining values of elliptic functions. Indeed, suppose that the base S is $\mathrm{Spec}(R)$, where R is the ring of integers of a number field. Then, as it was first observed in [A1], the statement of Theorem A with S as above, is equivalent to the vanishing of the so-called class-invariant homomorphism on

Received 12 August 1996. Revision received 1 November 1996.